Sharp operator-norm asymptotics for linearised elastic plates with rapidly oscillating periodic properties

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Outline

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- Problem formulation;
- Important estimates;
- Asymptotic procedure;
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Introduction

We look the problem

$$-\operatorname{div}(A(\frac{x}{\varepsilon})\nabla u) + u = f,$$

on $\mathbf{R}^n$, $f \in L^2(\mathbf{R}^n)$, $u \in H^1(\mathbf{R}^n)$. A is assumed to be one periodic symmetric, positive definite and bounded, i.e., there exist $\alpha, \beta > 0$

$$\alpha|\xi|^2 \leq A(x)\xi \cdot \xi \leq \beta|\xi|^2, \quad \forall x, \xi \in \mathbf{R}^n.$$

$$\mathcal{A}^\varepsilon u := -\operatorname{div} A(\frac{x}{\varepsilon})\nabla u.$$

Note that

$$(\mathcal{A}^\varepsilon + I)^{-1} : L^2(\mathbf{R}^n) \to L^2(\mathbf{R}^n)$$

is a continuous operator.

Introduction

The limit problem is given by (homogenization theory)

$$-\operatorname{div}(A^0 \nabla u) + u = f,$$

$$A^0 \xi \cdot \xi = \min_{\varphi \in H^1(\mathcal{Y})} \int_{\mathcal{Y}} A(y)(\xi + \nabla \varphi) \cdot (\xi + \nabla \varphi),$$

where $\mathcal{Y}$ is a flat torrus in $\mathbf{R}^n$.

$$\mathcal{A}^0 = -\operatorname{div}(A^0 \nabla u).$$

Claim: If u^ε is a solution of ε-problem that corresponds to f^ε and if $f^\varepsilon \rightharpoonup f$ in L^2, then the $u^\varepsilon \rightharpoonup u$ in H^1, where u is a solution of zero problem.

The result can be quantified.

Introduction

In a series of papers Birman, Suslina proved:

$$\|(\mathcal{A}^\varepsilon + I)^{-1} - (\mathcal{A} + I)^{-1}\|_{L^2(\mathbf{R}^n)\to L^2(\mathbf{R}^n)} \leq C\varepsilon,$$

$$\|(\mathcal{A}^\varepsilon + I)^{-1} - (\mathcal{A} + I)^{-1} - \varepsilon K(\varepsilon)\|_{L^2(\mathbf{R}^n)\to H^1(\mathbf{R}^n)} \leq C\varepsilon,$$

where $K(\varepsilon)$ is a corrector.
Later this was used for the estimates on finite domain by Suslina (2012), when one needs to include estimates of boundary layer (previous works with less sharp estimate by Zhikov, Pastukhova by Steklov smoothing, 2005, Griso by unfolding 2004).

Resolvent norm approximation in high contrast (Cherednichenko, Cooper)

We explain the approach of Cherednichenko, Cooper (ARMA 2016) in the context of high contrast.

$$-\operatorname{div}(A(\frac{x}{\varepsilon})\nabla u) + u = f,$$

on $\mathbf{R}^n$, $f \in L^2(\mathbf{R}^n)$, $u \in H^1(\mathbf{R}^n)$. A is assumed to be 1-periodic:

$$A = \chi_1 A_1 + \chi_0 \varepsilon^2 A_0 \text{ on } \mathcal{Y},$$

where χ_0 is a characteristic function of e.g. ball $B \subset (0,1)^n$, $\chi_1 = 1 - \chi_0$, A_0, A_1 are symmetric, uniformily elliptic.
The qualitative analysis of these kind of operators was given by Zhikov. The limit operator is defined on the subspace of $L^2(\mathbf{R}^n \times \mathcal{Y})$ and its spectrum has band gap structure.

Resolvent norm approximation in high contrast (Cherednichenko, Cooper)

Cherednichenko, Cooper found the operator $\tilde{\mathcal{A}}^\varepsilon$ which is simpler then the starting one and which satisfies

$$\|(\mathcal{A}^\varepsilon + I)^{-1} - (\tilde{\mathcal{A}}^\varepsilon + I)^{-1}\tilde{\mathcal{P}}^\varepsilon\|_{L^2(\mathbf{R}^n)\to L^2(\mathbf{R}^n)} \leq C\varepsilon.$$

The operator $\tilde{\mathcal{A}}^\varepsilon$ is still ε- dependent and $\tilde{\mathcal{P}}^\varepsilon$ is a kind of projection. Resolvent approximation implies the approximation of spectrum of the operator $\mathcal{A}^\varepsilon$ in the Hausdorff sense. The operator $\tilde{\mathcal{A}}^\varepsilon$ differs from the limit operator obtained by qualitative analysis of Zhikov, since it contains more information. However, it is still computionally much cheaper than the original one. The method offers the way not only to prove the estimates, but also to change (or slightly perturb) the expected approximate operator.

Resolvent norm approximation in high contrast (Cherednichenko, Cooper)

$$Q' = [-\pi, \pi)^n, \quad Q = [0, 1)^n.$$

We define the isometry $\mathcal{U}^\varepsilon : L^2(\mathbf{R}^n) \to L^2\big(\varepsilon^{-1} Q' \times Q\big)$ by

$$(\mathcal{U}^\varepsilon f)(\theta, y) \;=\; \left(\frac{\varepsilon^2}{2\pi}\right)^{n/2} \sum_{n \in \mathbf{Z}^n} f\big(\varepsilon(y+n)\big) \exp\big(-i\varepsilon\theta(y+n)\big),$$

$$\theta \in \varepsilon^{-1} Q', \quad y \in Q.$$

This isometry $\mathcal{U}^\varepsilon = \mathcal{G}^\varepsilon \mathcal{T}^\varepsilon$ is a composition of usual scaled Gelfand transform $\mathcal{G}^\varepsilon : L^2(\mathbf{R}^n) \to L^2\big(\varepsilon^{-1} Q' \times \varepsilon Q\big)$:

$$(\mathcal{G}^\varepsilon f)(\theta, z) = \left(\frac{\varepsilon}{2\pi}\right)^{n/2} \sum_{n \in \mathbf{Z}^n} f\big(z + \varepsilon n\big) \exp\big(-i\varepsilon\theta(y+n)\big), \quad z \in \varepsilon Q,$$

and the scaling transform $\mathcal{T}^\varepsilon : L^2(\varepsilon^{-1} Q' \times \varepsilon Q) \to L^2(\varepsilon^{-1} Q' \times Q)$

$$(\mathcal{T}^\varepsilon h)(\theta, y) = \varepsilon^{n/2} h(\theta, \varepsilon y).$$

Resolvent norm approximation in high contrast (Cherednichenko, Cooper)

Then we have that

$$\mathcal{U}^{\varepsilon}(\mathcal{A}^{\varepsilon}+I)^{-1}(\mathcal{U}^{\varepsilon})^{-1}=\int_{\varepsilon^{-1}Q'}^{\oplus}(\mathcal{B}_{\varepsilon,\theta}+I)^{-1}d\theta,$$

where $\mathcal{B}_{\varepsilon,\theta}$ is the operator generated by the sesquilinear form

$$b^{\varepsilon,\theta}(u,v)=\int_{Q}(\varepsilon^{-2}A_1+A_0)(\nabla+i\varepsilon\theta)u\cdot\overline{(\nabla+i\varepsilon\theta)v},\quad u,v\in H^1_{\#}(Q)$$

or

$$\tilde{b}^{\varepsilon,\theta}(u,v)=\int_{Q}(\varepsilon^{-2}A_1+A_0)\nabla u\cdot\overline{\nabla v},\quad u,v\in H^1_{\chi}(Q),\chi=\varepsilon\theta.$$

Resolvent norm approximation in high contrast (Cherednichenko, Cooper)

For every parameter $\theta \in \varepsilon^{-1}Q'$ we obtain a differential equation on a compact domain Q (the equation can be looked with periodic or quasi-periodic boundary condition). This is a standard approach for periodic problems. In this way one can divide the spectrum of the original operator (on a non-compact domain) as a union of continuum family of spectrum of operators on a compact domain. One can even characterize generalized eigenfunctions of the original operator.
The novelty of the approach of Cherednichenko and Cooper consists in finding the operators $\mathcal{B}_{\mathrm{hom}}^{\varepsilon,\theta}$ such that

$$\|(\mathcal{B}^{\varepsilon,\theta}+I)^{-1}-(\mathcal{B}_{\mathrm{hom}}^{\varepsilon,\theta}+I)^{-1}\mathcal{P}^{\varepsilon}\|_{L^2(Q)\to L^2(Q)} \leq C\varepsilon,$$

where C is independent of θ and $\mathcal{P}^{\varepsilon}$ is a projection.

Resolvent norm approximation in high contrast (Cherednichenko, Cooper)

The methodology consists in doing formal asymptotics for the solution of the equation

$$u_\theta^\varepsilon = \sum_{n=0}^{\infty} \varepsilon^n u_\theta^{(n)}, \quad u_\theta^{(n)} \in H^1_\#(Q).$$

plugging it into equation and obtaining the approximate solution. The difficulties arise in the fact that one has to do the estimates and the fact that there are two changing parameters (ε and θ) and that there are no rules (ansatz) how to do this asymptotics in the way to obtain the estimates. They had to analyze separately so called inner region ($|\theta| \leq 1$), intermediate region $1 \leq |\theta| \leq \varepsilon^{-1/2}$) and upper region $|\theta| \geq \varepsilon^{-1/2}$. In the case without high contrast upper region can be neglected, i.e., the good approximation of the solution is zero.

Problem formulation

$$\int_{\Omega^h} A\left(\frac{x_1}{\varepsilon}, \frac{x_2}{\varepsilon}\right) \mathrm{sym}\nabla u^h : \mathrm{sym}\nabla \phi^h + \int_{\Omega^h} u^h \cdot \phi^h = \int_{\Omega^h} f^h \cdot \phi^h.$$

Here

$$\Omega^h = \mathbf{R}^2 \times (-h/2, h/2),\ u^h, \phi^h \in H^1(\Omega^h, \mathbf{R}^3),\ f^h \in L^2(\Omega^h, \mathbf{R}^3).$$

A is one periodic, bounded and coercive on symmetric matrices, i.e., there exist $\alpha, \beta > 0$ such that

$$\alpha |M|^2 \le A(y_1, y_2) M \cdot M \le \beta |M|^2, \quad \forall (y_1, y_2) \in [0,1]^2, M \in \mathbf{R}^{3\times 3}_{\mathrm{sym}}.$$

Problem formulation

We transform the problem on the domain $\Omega = \mathbf{R}^2 \times I$, $I = (-\frac{1}{2}, \frac{1}{2})$ by doing change $x_3 = x_3^h/h$. We do the Gelfand transform and scale in-plane components $y_\alpha = x_\alpha/\varepsilon$, for $\alpha = 1, 2$. After that we obtain:

$$\int_Q A(y)\,\mathrm{sym}\left(\tilde{\nabla}^{\varepsilon,h,\theta}(u_1, u_2, u_3)\right) : \overline{\mathrm{sym}\left(\tilde{\nabla}^{\varepsilon,h,\theta}(\varphi_1, \varphi_2, \varphi_3)\right)} + \int_Q u_\alpha \overline{\varphi}_\alpha + \int_Q u_3 \overline{\varphi}_3 = \int_Q f_i\,\overline{\varphi}_i, \quad \forall \varphi \in H^1_\#(Q, \mathbf{C}^3).$$

Here

$$Q = Q_r \times I,\ Q_r = (0,1)^2,\ \theta \in \varepsilon^{-1}(-\pi, \pi)^2.$$

$$(\tilde{\nabla}^{\varepsilon,h,\chi} v)_{i\alpha} := \frac{1}{\varepsilon}(\partial_\alpha + i\chi_\alpha) v_i,$$

$$(\tilde{\nabla}^{\varepsilon,h,\chi} v)_{i3} := \frac{1}{h}\partial_3 v_i, \quad \alpha = 1, 2,\ i = 1, 2, 3.$$

Problem formulation

We will look the regime $\varepsilon = h$. We obtain:

$$\frac{1}{\varepsilon^2}\int_Q A(y)\,\mathrm{sym}\tilde{\nabla}_{y,x_3}(u_1,u_2,u_3) : \overline{\mathrm{sym}\tilde{\nabla}_{y,x_3}(\varphi_1,\varphi_2,\varphi_3)}$$
$$+\int_Q u_\alpha\overline{\varphi_\alpha}+\int_Q u_3\overline{\varphi_3}=\int_Q f_\alpha\overline{\varphi_\alpha}+\int_Q f_3\overline{\varphi_3}, \quad \forall\varphi\in H^1_\#(Q,\mathbf{C}^3).$$

Here

$$(\tilde{\nabla}u)_{i\alpha}=\partial_\alpha u_i+i\chi_\alpha u_i,\ (\tilde{\nabla}u)_{i3}=\partial_3 u_i,\ \alpha=1,2,\ i=1,2,3.$$

The equations can be looked on the space of χ- quasiperiodic functions $H^1_\chi(Q,\mathbf{C}^3)$ in which case we replace $\tilde{\nabla}$ by ∇.

Limit equation on finite domain

Ciarlet and Kesavan (1981) looked the spectral problem on the bounded domain $\Omega = \omega \times I$, where $\omega \subset \mathbf{R}^2$ is open bounded set with Lipschitz boundary in the case of isotropic homogeneous plate (clamped plate). After scaling the forces in the way $f^h = (h^2 f_1, h^2 f_2, h^3 f_3)$, the displacement in the way $u^h = (h^2 u_1, h^2 u_2, h u_3)$ and the density (spectrum) with h^2 in the limit they obtained the folowing problem: find $u_3 \in H^2(\omega, \mathbf{R}^3)$, $u_3 = \partial_n u_3 = 0$ on $\partial\omega$ that satisfies

$$\int_\omega \left(\frac{4\lambda\mu}{3(\lambda + 2\mu)} \Delta u_3 \Delta v_3 + \frac{4\mu}{3} \partial_{\alpha\beta} u_3 \partial_{\alpha\beta} v_3 \right) = 2\Lambda \int_\omega u_3 v_3,$$

$$v_3 \in H^2(\omega, \mathbf{R}^3),\ v_3 = \partial_n v_3 = 0 \text{ on } \partial\omega.$$

Limit equation on finite domain

Comments:

- It is proved that n-th eigenvalue of h problem (after scaled with h^2) converges to the n-th eigenvalue of the limit problem. The similar claim is valid for eigenfunctions.
- The limit equation is the spectral problem of forth order for the vertical displacement. It is proved that eigenvalues of the original problem converge to the eigenvalues of this limit problem in the Hausdorff sense;
- The fact that we obtain the limit problem only for u_3 is the consequence of scaling for the density and the fact that in the limit problem (in the case of isotropic media) the equations for the vertical displacement separate from the equations of horizontal displacements;
- For eigenfunctions the limit horizontal displacement is $u_1 = -x_3 \partial_1 u_3$, $u_2 = -x_3 \partial_2 u_3$.

Limit equation on finite domain

- Quantitative estimates are proved by Dauge, Djurdjević, Faou, Rössle (1999). They divided the problem into two invariant subspaces, in one subspace there is the spectrum of order h^2 in other subspace the spectrum of order one. They again prove that the n-th eigenvalue of h problem is in relative norm distanced by h from the n-th eigenvalue of the limit problem. They also prove that the eigenfunctions are at distance h from limit eigenfunction.

Important estimates-Korn type inequalities

By using Korn's inequality and boundary condition we can show that for $u \in H^1_\chi(Q, \mathbf{C}^3)$ we have

$$\begin{aligned}
\left\| u_1 - (c_1 - i\chi_1 c_3 x_3) e_\chi(y) \right\|_{H^1(Q)} &\lesssim \|\mathrm{sym}\nabla u\|_{L^2(Q)}, \\
\left\| u_2 - (c_2 - i\chi_2 c_3 x_3) e_\chi(y)) \right\|_{H^1(Q)} &\lesssim \|\mathrm{sym}\nabla u\|_{L^2(Q)}, \\
\left\| u_3 - c_3 e_\chi(y) \right\|_{H^1(Q)} &\lesssim \|\mathrm{sym}\nabla u\|_{L^2(Q)},
\end{aligned}$$

for some $c_1, c_2, c_3 \in \mathbf{C}$ which satisfy

$$\begin{aligned}
\max\{|c_1|, |c_2|\} &\lesssim \frac{1}{|\chi|} \|\mathrm{sym}\nabla u\|_{L^2(Q)}, \\
|c_3| &\lesssim \frac{1}{|\chi|^2} \|\mathrm{sym}\nabla u\|_{L^2(Q)}.
\end{aligned}$$

Here $e_\chi(y) = \exp(\mathrm{i}\chi \cdot y)$, $y \in Q$. The following is satisfied

$$\begin{aligned}
\chi \neq 0, \mathrm{sym}\,\nabla u = 0 \quad &\Longrightarrow \quad u = 0, \\
\chi = 0, \mathrm{sym}\,\nabla u = 0 \quad &\Longrightarrow \quad u = Ax + b(\text{ with periodicity } A = 0), \\
& \qquad\;\; A \in \mathbf{C}^{3\times 3}_{\mathrm{skew}}, b \in \mathbf{C}^3.
\end{aligned}$$

Important estimates-Korn type inequalities

We additionally assume “planar" material symmetries:

$$A_{\alpha\beta\gamma 3} = 0, \qquad A_{\alpha 333} = 0 \qquad \forall \alpha, \beta, \gamma \in \{1, 2\}.$$

Under this assumption we have two invariant subspaces of the elasticity operator:

1. horizontal displacements odd in x_3 variable and vertical displacement even in x_3 variable;
2. horizontal displacement even in x_3 variable and vertical displacement odd in x_3 variable.

Important estimates-Korn type inequalities

In the first subspace we have that the lowest eigenvalue is of order (at least) $|\chi|^4$ and we have the estimates

$$
\begin{aligned}
\left\|u_1 + i\chi_1 c_3 x_3 e_\chi(y)\right\|_{H^1(Q)} &\lesssim \|\mathrm{sym}\nabla u\|_{L^2(Q)}, \\
\left\|u_2 + i\chi_2 c_3 x_3 e_\chi(y)\right\|_{H^1(Q)} &\lesssim \|\mathrm{sym}\nabla u\|_{L^2(Q)}, \\
\left\|u_3 - c_3 e_\chi(y)\right\|_{H^1(Q)} &\lesssim \|\mathrm{sym}\nabla u\|_{L^2(Q)}.
\end{aligned}
$$

In the second subspace we have that the lowest eigenvalue is of order (at least) $|\chi|^2$ and we have the estimates

$$
\begin{aligned}
\left\|u_1 - c_1 e_\chi(y)\right\|_{H^1(Q)} &\lesssim \|\mathrm{sym}\nabla u\|_{L^2(Q)}, \\
\left\|u_2 - c_2 e_\chi(y)\right\|_{H^1(Q)} &\lesssim \|\mathrm{sym}\nabla u\|_{L^2(Q)}, \\
\left\|u_3\right\|_{H^1(Q)} &\lesssim \|\mathrm{sym}\nabla u\|_{L^2(Q)}.
\end{aligned}
$$

These estimates can be interpreted as spectral gap estimates.

Towards asymptotics

In the first subspace we will scale the operator with $\frac{1}{|\chi|^4}$ and scale the horizontal forces with $\frac{1}{|\chi|}$. The equation becomes

$$\frac{1}{|\chi|^4}\int_Q A\,\mathrm{sym}\nabla_{y,x_3}(u_1,u_2,u_3):\overline{\mathrm{sym}\nabla_{y,x_3}(\varphi_1,\varphi_2,\varphi_3)}+$$

$$\int_Q u_\alpha\overline{\varphi_\alpha}+\int_Q u_3\overline{\varphi_3}=\frac{1}{|\chi|}\int_Q f_\alpha\overline{\varphi_\alpha}+\int_Q f_3\overline{\varphi_3}\qquad \forall\varphi\in H^1_\chi(Q,\mathbf{C}^3).$$

We assume that f_α is odd in x_3 variable, while f_3 is even. As a consequence of material symmetries we have that the dispacement satisfies the same conditions.

Towards asymptotics

In the second subspace we will scale the operator with $\frac{1}{|\chi|^2}$.
The equation becomes

$$\frac{1}{|\chi|^2}\int_Q A\,\mathrm{sym}\nabla_{y,x_3}(u_1,u_2,u_3) : \overline{\mathrm{sym}\nabla_{y,x_3}(\varphi_1,\varphi_2,\varphi_3)} +$$

$$\int_Q u_\alpha \overline{\varphi_\alpha} + \int_Q u_3\overline{\varphi_3} = \int_Q f_\alpha\overline{\varphi_\alpha} + \int_Q f_3\overline{\varphi_3} \qquad \forall\varphi\in H^1_\chi(Q,\mathbb{C}^3).$$

Here we assumed that f_α is even in x_3 variable, while f_3 is odd.
As a consequence of material symmetries we have that the displacement satisfies the same conditions.

Towards asymptotics

- Apriori estimates are telling us that there is at most one-dimensional eigenspace in the first invariant subspace and two-dimensional eigenspace in the second invariant subspace where the order of eigenvalue is less than one.
- The approximate equation will tell us that we caught all eigenvalues whose order is less than one.
- This will provide us more information on the spectrum than the norm resolvent estimate (the norm resolvent estimate by its nature does not provide good estimate for large spectrum and small spectrum)
- Our estimate implies that if we scale the original operator with ε^{α}, $\alpha > 0$, then the approximate equation has precision (wrt norm resolvent) $\varepsilon^{\alpha/4}$ in the first subspace and $\varepsilon^{\alpha/2}$ in the second.

Towards asymptotics

- We will do all estimates depending on parameter χ (not on ε). In the first subspace we will have the spectrum of order $|\chi|^4$, in the second of order $|\chi|^2$ (for the original operator $\frac{|\chi|^4}{\varepsilon^2}$, i.e., $\frac{|\chi|^2}{\varepsilon^2}$). Moreover the eigenfunctions in the first invariant subspace have in-plane components of order $|\chi|$, in the second subspace the order of vertical component of eigenfunction will be $|\chi|$. Our approximate equation will approximate the spectrum with precision $|\chi|$ in relative norm and also the eigenfunctions in the first subspace with relative precision $|\chi|$. In the second subspace we will approximate the eigenfunctions with precision $|\chi|$. Better precision would require different approximate equation.

Towards asymptotics

- Scaling of the forces is connected with the additional precision we want to have on first two components in the first subspace (they are of order $|\chi|$, we obtain precision of order $|\chi|^2$). This can be proved by Riesz representation theorem.
- This natural scaling has the consequence that the computations are more elegant than they would be with scaling depending on ε! By scaling the operator we don't have anymore to divide the problem into the inner, intermediate and upper region. This is natural since there is no anymore unity. The only area where these estimates do not provide any information is the region $|\chi| \sim 1$ (recall the work of Allaire, Conca).

Apriori estimates

Apriori estimates in the first subspace:

$$\begin{aligned}\left\|\mathrm{sym}\,\nabla u\right\|_{L^2(Q,\mathbf{C}^{3\times 3})} &\lesssim |\chi|^2\|f\|_{L^2(Q,\mathbf{C}^3)},\\ \|u_1\|_{L^2(Q,\mathbf{C})}+\|u_2\|_{L^2(Q,\mathbf{C})} &\lesssim |\chi|\|f\|_{L^2(Q,\mathbf{C}^3)},\\ \|u_3\|_{L^2(Q,\mathbf{C})} &\lesssim \|f\|_{L^2(Q,\mathbf{C}^3)}.\end{aligned}$$

The approximate equation will approximate the solution in in-plane components with order $|\chi|^2$, while in vertical component with order $|\chi|$. This will be valid also under scaling of in-plane forces with order $\frac{1}{|\chi|}$.

Apriori estimates

Apriori estimates in the second subspace:

$$\begin{aligned} \left\|\mathrm{sym}\,\nabla u\right\|_{L^2(Q,\mathbf{C}^{3\times 3})} &\lesssim C|\chi|\|f\|_{L^2(Q,\mathbf{C}^3)}; \\ \|u_1\|_{H^1(Q,\mathbf{C})} + \|u_2\|_{H^1(Q,\mathbf{C})} &\lesssim \|f\|_{L^2(Q,\mathbf{C}^3)}, \\ \|u_3\|_{H^1(Q,\mathbf{C})} &\lesssim |\chi|\|f\|_{L^2(Q,\mathbf{C}^3)}. \end{aligned}$$

If we take $f_\alpha = 0$, for $\alpha = 1, 2$ we obtain:

$$\begin{aligned} \left\|\mathrm{sym}\,\nabla u\right\|_{L^2(Q,\mathbf{C}^{3\times 3})} &\lesssim |\chi|^2\|f_3\|_{L^2(Q,\mathbf{C})}, \\ \|u_1\|_{H^1(Q,\mathbf{C})} + \|u_2\|_{H^1(Q,\mathbf{C})} &\lesssim |\chi|\|f_3\|_{L^2(Q,\mathbf{C})}, \\ \|u_3\|_{H^1(Q,\mathbf{C})} &\lesssim |\chi|^2\|f_3\|_{L^2(Q,\mathbf{C})}. \end{aligned}$$

We will show that the approximate equation approximates the original one (wrt scaled norm resolvent) with order $|\chi|$ and thus we can neglect f_3.

Approximate equation

The limit equation in the first subspace has the form

$$\frac{1}{|\chi|^4} A^{\mathrm{hom},1} m_3 \, \overline{d_3} +$$
$$\int_Q (-i\chi_1 x_3 m_3, -i\chi_2 x_3 m_3, m_3)^\top \cdot \overline{(-i\chi_1 x_3 d_3, -i\chi_2 x_3 d_3, d_3)^\top}$$
$$= \frac{1}{|\chi|} \int_Q (f_1, f_2)^\top \cdot \overline{e_\chi (-i\chi_1 x_3 d_3, -i\chi_2 x_3 d_3)^\top}$$
$$+ \int_Q f_3 \cdot \overline{e_\chi d_3}, \qquad \forall d_3 \in \mathbf{C}.$$

Approximate equation

We define

$$\Upsilon(\chi, m_3) := i m_3 \begin{pmatrix} \chi_1^2 & \chi_1\chi_2 \\ \chi_1\chi_2 & \chi_2^2 \end{pmatrix};$$

$$\begin{aligned} (\mathrm{sym}\,\nabla)^* A\,\mathrm{sym}\,\nabla N_m^{(1)} &= (\mathrm{sym}\,\nabla)^* A\, i x_3 \Upsilon(\chi, m_3), \\ & N_m^{(1)} \in H^1_\#(Q, \mathbf{C}^3), \quad \int_Q N_m^{(1)} = 0; \end{aligned}$$

$$A^{\mathrm{hom},1} m_3\, \overline{d_3} := \int_Q A\big(\nabla N_m^{(1)} - i x_3 \Upsilon(\chi, m_3)\big) : \overline{\big(-i x_3 \Upsilon(\chi, d_3)\big)}.$$

It can be shown that the lower order terms on the left hand side can be neglected.

Approximate equation

The limit equation in the second subspace has the form

$$\frac{1}{|\chi|^2} A^{\text{hom},2}(m_1, m_2)^\top \cdot \overline{(d_1, d_2)^\top} + (m_1, m_2)^\top \cdot \overline{(d_1, d_2)^\top} =$$
$$\int_Q (f_1, f_2)^\top \cdot \overline{e_\chi (d_1, d_2)^\top} \qquad \forall (d_1, d_2)^\top \in \mathbf{C}^2.$$

We define

$$\Xi(\chi, m_1, m_2) := i \begin{pmatrix} \chi_1 m_1 & \frac{1}{2}(\chi_1 m_2 + \chi_2 m_1) \\ \frac{1}{2}(\chi_1 m_2 + \chi_2 m_1) & \chi_2 m_2 \end{pmatrix};$$

$$(\operatorname{sym} \nabla)^* A \operatorname{sym} \nabla N_m^{(2)} = -(\operatorname{sym} \nabla)^* A \Xi(\chi, m_1, m_2),$$
$$N_m^{(2)} \in H^1_\#(Q, \mathbf{C}^3), \quad \int_Q N_m^{(2)} = 0;$$

$$A^{\text{hom},2}(m_1, m_2)^\top \cdot \overline{(d_1, d_2)^\top} := \int_Q A\big(\nabla N_m^{(2)} + \Xi(\chi, m_1, m_2)\big) : \overline{\Xi(\chi, d_1, d_2)}.$$

Asymptotics in the second subspace

The following estimate is satisfied for the solution of the limit equation

$$\left|(m_1, m_2)\right| \lesssim \left\|(f_1, f_2)\right\|_{L^2(Q,\mathbf{C}^2)}.$$

The asymptotic procedure starts with definition $\mathfrak{u}_2 \in H^1_\#(Q, \mathbf{C}^3)$

$$(\operatorname{sym}\nabla)^* A \operatorname{sym}\nabla\mathfrak{u}_2 = -(\operatorname{sym}\nabla)^* A\Xi(\chi, m_1, m_2), \qquad \int_Q \mathfrak{u}_2 = 0.$$

We infer that

$$\|\mathfrak{u}_2\|_{H^1(Q,\mathbf{C}^3)} \lesssim |\chi|\left\|(f_1, f_2)\right\|_{L^2(Q,\mathbf{C}^2)}.$$

Asymptotics in the second subspace

Next we define $\mathfrak{u}_3 \in H^1_\#(Q, \mathbf{C}^3)$ that satisfies $\int_Q \mathfrak{u}_3 = 0$ and

$$\begin{aligned}(\mathrm{sym}\,\nabla)^* A\,\mathrm{sym}\,\nabla \mathfrak{u}_3 \quad = \quad & i\big\{\mathcal{O}^* A\,\mathrm{sym}\,\nabla \mathfrak{u}_2 - (\mathrm{sym}\,\nabla)^* A\mathcal{O}\mathfrak{u}_2 \\ & + \mathcal{O}^* A\Xi(\chi, m_1, m_2)\big\} \\ & -|\chi|^2(m_1, m_2, 0)^\top + |\chi|^2\overline{e_\chi}(f_1, f_2, 0)^\top.\end{aligned}$$

Here

$$\mathcal{O}\varphi := \begin{pmatrix} \chi_1\varphi_1 & \frac{1}{2}(\chi_2\varphi_1 + \chi_1\varphi_2) & \frac{1}{2}\chi_1\varphi_3 \\ \frac{1}{2}(\chi_2\varphi_1 + \chi_1\varphi_2) & \chi_2\varphi_2 & \frac{1}{2}\chi_2\varphi_3 \\ \frac{1}{2}\chi_1\varphi_3 & \frac{1}{2}\chi_2\varphi_3 & 0 \end{pmatrix}.$$

Asymptotics in the second subspace

The right-hand side of the definition eqution for $\mathfrak{u}_3$ yields zero when tested with constant vectors of the form $(D_1, D_2, 0)^\top$ as the consequence of definition of $\mathfrak{u}_2$. Furthermore, it also vanishes when tested with vectors of the form $(0, 0, D_3)^\top$, as a consequence of symmetry conditions. Thus the equation for $\mathfrak{u}_3$ has the solution and we have the estimate:

$$\|\mathfrak{u}_3\|_{H^1(Q,\mathbf{C}^3)} \lesssim |\chi|^2 \big\|(f_1, f_2)\big\|_{L^2(Q,\mathbf{C}^3)}.$$

We stop with the asymptotics. We define the approximate solution

$$U = (m_1, m_2, 0)^\top + \mathfrak{u}_2 + \mathfrak{u}_3,$$

which clearly satisfies

$$(\mathrm{sym}\,\nabla + i\mathcal{O})^* A(\mathrm{sym}\,\nabla + i\mathcal{O})U + |\chi|^2 U = |\chi|^2 \overline{e_\chi}(f_1, f_2, 0)^\top + O(|\chi|^3),$$

where the remainder is estimated:

$$\big\|O(|\chi|^3)\big\|_{H^{-1}_{\#}(Q,\mathbf{C}^3)} \lesssim |\chi|^3 \big\|(f_1, f_2)\big\|_{L^2(Q,\mathbf{C}^2)}.$$

Asymptotics in the second subspace

It follows that the error $z := (u_1, u_2, u_3) - U$ satisfies

$$(\mathrm{sym}\,\nabla + i\mathcal{O})^* A(\mathrm{sym}\,\nabla + i\mathcal{O})z + |\chi|^2 z = O(|\chi|^3).$$

It is easy to see that

$$\big\|A(\mathrm{sym}\,\nabla + i\mathcal{O})z\big\|^2_{L^2(Q,\mathbf{C}^{3\times 3})} \gtrsim \|\,\mathrm{sym}\,\nabla z\|^2_{L^2(Q,\mathbf{C}^{3\times 3})} - |\chi|\|z\|^2_{L^2(Q,\mathbf{C}^3)},$$

for some $0 < c_1 < 1$ and thus, for sufficiently small ε (recall $|\chi| \ll 1$), one has

$$\big\|A(\mathrm{sym}\,\nabla + i\mathcal{O})z\big\|^2_{L^2(Q,\mathbf{C}^{3\times 3})} + |\chi|^2\|z\|^2_{L^2(Q,\mathbf{C}^3)} \gtrsim |\chi|^2\big(\|\,\mathrm{sym}\,\nabla z\|^2_{L^2(Q,\mathbf{C}^{3\times 3})} + \|z\|^2_{L^2(Q,\mathbf{C}^3)}\big).$$

By testing the equation with z we conclude

$$\|z\|_{H^1(Q,\mathbf{C}^3)} \lesssim |\chi|\|(f_1, f_2)\|_{L^2(Q,\mathbf{C}^2)}.$$

Asymptotics in the second subspace

From the estimates we have on $\mathfrak{u}_2$, $\mathfrak{u}_3$ we conclude for the solution of the scaled equation

$$\begin{aligned} \|u_\alpha - m_\alpha e_\chi\|_{H^1(Q,\mathbf{C})} &\lesssim |\chi| \|(f_1, f_2)\|_{L^2(Q,\mathbf{C}^2)}, \quad \alpha = 1,2, \\ \|u_3\|_{H^1(Q,\mathbf{C})} &\lesssim |\chi| \big\|(f_1, f_2)\big\|_{L^2(Q,\mathbf{C}^2)}. \end{aligned}$$

Further remarks on asymptotics

- In the first invariant subspace we have to go further into the expansion to obtain the satisfied precision. The obtained estimates are (recall that the forces are scaled)

$$\begin{aligned} \|u_\alpha + i\chi_\alpha m_3 x_3 e_\chi\|_{H^1(Q,\mathbf{C})} &\lesssim |\chi|^2 \| \|f\|_{L^2(Q,\mathbf{C}^3)}, \quad \alpha = 1, 2, \\ \|u_3 - m_3 e_\chi\|_{H^1(Q,\mathbf{C})} &\lesssim |\chi| \| \|f\|_{L^2(Q,\mathbf{C}^3)}. \end{aligned}$$

- If we do not assume the planar symmetries we can easily do the computations when we scale the operator with ε^2 (usual norm resolvent estimate). However, by now, I don't know how to do the computations with the scaling ε^4;
- It can be easily seen that the approximate equation has the spectrum of order one. By standard argument with min max principle and Rayleigh quotient we can prove that with the approximate equation we approximate the spectrum with precision $|\chi|$. For approximating the spectrum we don't need to scale the forces in the first invariant subspace.

Further remarks on asymptotics

- Scaling of the forces is used to obtain the precision of order $|\chi|^2$ in the in-plane components of eigenfunctions in the first invariant subspace (they themselves have the order $|\chi|$). The argument goes by Riesz representation theorem.
- Although the approximate equation in the first subspace does not have the resolvent form in the strict sense, it resembles resolvent form and can be used in estimations (for min max principle and Riesz representation formula).
- Approximate equation does not contain the relevant approximation only for $|\chi| \sim 1$, since the error is of the same order as the spectrum. Moreover, in order to analyze the spectrum of order one, one would have to include this upper region as well as more eigenvalues in the inner region. Usual norm resolvent approximation neglects the area where $|\chi| \sim \sqrt{\varepsilon}$. Moreover, for spectrum of order less than ε it does not provide satisfactory precision.

Further remarks on asymptotics

- Considering wave propagation, interest for approximating the scaled operator (scaling with ε^{α}) in the norm resolvent sense arises if one would like to approximate waves with certain prescribed level of energy (again scaled with the power of ε).

Limit equation

Bending equation (first subspace):

$$\frac{1}{12}\int_{\mathbf{R}^2} \mathcal{L}\nabla^2 v : \nabla^2\psi + \frac{\varepsilon^2}{12}\int_{\mathbf{R}^2} \nabla v \cdot \nabla\psi + \int_{\mathbf{R}^2} v\psi =$$
$$-\int_{\mathbf{R}^2} \Big(\big\langle x_3 f_1\big\rangle, \big\langle x_3 f_2\big\rangle\Big)^\top \cdot \nabla\psi + \int_{\mathbf{R}^2} \langle f_3\rangle\psi \qquad \forall\psi \in H^2(\mathbf{R}^2, \mathbf{R}),$$

$$\mathcal{L}M : M \quad := \quad \inf_{\psi\in H^1_\#(Q,\mathbf{R}^3)} \int_Q A\big(\iota(M) + \mathrm{sym}\,\nabla\psi\big) : \big(\iota(M) + \mathrm{sym}\,\nabla\psi\big),$$
$$M \in \mathbf{R}^{2\times 2}_{\mathrm{sym}},\ \iota : \mathbf{R}^{2\times 2}_{\mathrm{sym}} \to \mathbf{R}^{3\times 3}_{\mathrm{sym}},\ \text{natural inclusion.}$$

$$\langle f\rangle := \int_I f\, dx_3.$$

The displacement is approximated with $(-\varepsilon x_3\partial_1 v, -\varepsilon x_3\partial_2 v, v)$.

Limit equation

Stretching equation (second subspace):

$$\int_{\mathbf{R}^2} \mathcal{L} \,\mathrm{sym}\, \nabla w : \mathrm{sym}\, \nabla \psi + \int_{\mathbf{R}^2} w \cdot \psi \;=\; \int_{\mathbf{R}^2} \Big(\langle f_1 \rangle, \langle f_2 \rangle \Big)^\top \cdot \psi, \quad \forall \psi \in H^1(\mathbf{R}^2, \mathbf{R}^2).$$

The displacement is approximated with $(w_1, w_2, 0)$.

Thank you for your attention!