Quantitative homogenization in non-linear elasticity for small loads

Mathias Schäffner

joint work with Stefan Neukamm

19.06.2018

TECHNISCHE UNIVERSITÄT DRESDEN

DRESDEN concept

Non-linear elasticity model of periodic composite.

$$\mathcal{I}_\varepsilon(u_\varepsilon) = \int_\Omega W(\tfrac{x}{\varepsilon}, \nabla u_\varepsilon) - f \cdot u_\varepsilon \to \min! \qquad \text{subject to BC}$$

Homogenization limit.

$$\mathcal{I}_{\hom}(u_0) = \int_\Omega W_{\hom}(\nabla u_0) - f \cdot u_0 \to \min! \qquad \text{subject to BC}$$

$$W_{\hom}(F) := \inf_{k\in\mathbb{N}} \inf_{\phi \in W^{1,p}_{\mathrm{per}}(k\Box)} \fint_{k\Box} W(y, F + \nabla\phi(y))\,dy$$

Results of this talk:

- One-cell homogenization formula & existence of corrector ϕ_F

$$\operatorname{dist}(F, SO(d)) \ll 1 \quad \Rightarrow \quad W_{\hom}(F) = \fint_\Box W(y, F + \nabla\phi_F(y))\,dy.$$

- Quantitative two-scale expansion (for small data)
- Uniform Lipschitz estimates for minimizer (for small & well-prepared data)

Program

(i) Variational model for elasticity / basic homogenization results

(ii) Validity of the one-cell formula close to rotations

(iii) Quantitative two-scale expansion (for small data)

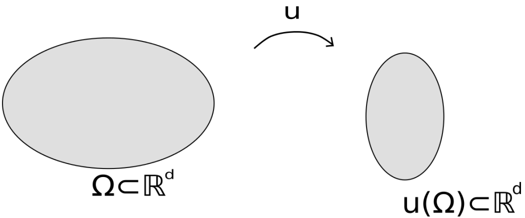
(iv) Uniform Lipschitz estimates (for small & well prepared data)

Variational model for non-linear elasticity

$$\mathcal{I}(u) := \int_\Omega W(\nabla u) - f \cdot u \qquad \text{(elastic energy functional)},$$

where W is a non-convex energy density, satisfying

- $W(F) = W(RF)\ \forall F \in \mathbb{R}^{d\times d},\ R \in \mathrm{SO}(d)$ (**frame indifferent**)
- $W(\mathrm{Id}) = \min W = 0$ (**reference configuration = natural state**)
- $W(F) \gtrsim \mathrm{dist}^2(F, \mathrm{SO}(d))\ \forall F \in \mathbb{R}^{d\times d}$ (**non-degeneracy**)

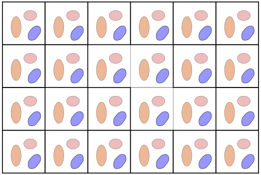


Variational model for non-linear elastic composite

$$\mathcal{I}_\varepsilon(u) := \int_\Omega W(\tfrac{x}{\varepsilon}, \nabla u) - f \cdot u \, dx \qquad \text{(elastic energy functional)},$$

where W is a non-convex energy density, satisfying

- $W(y,F) = W(y,RF)\ \forall F \in \mathbb{R}^{d\times d},\ R \in \mathrm{SO}(d)$ (**frame indifferent**)
- $W(y,\mathrm{Id}) = \min W = 0$ (**reference configuration = natural state**)
- $W(y,F) \gtrsim \mathrm{dist}^2(F,\mathrm{SO}(d))\ \forall F \in \mathbb{R}^{d\times d}$ (**non-degeneracy**)
- $W(y,F)$ is $\Box := [0,1)^d$ - periodic in y
- $0 < \varepsilon \ll 1$ size of the micro-structure



Homogenization of convex integral functionals

$$\mathcal{E}_\varepsilon(u) := \int_\Omega V(\tfrac{x}{\varepsilon}, \nabla u(x))\,dx$$

Suppose: $V(y,F)$ $\square$-periodic in y; convex & p-growth in F, i.e

$$c|F|^p - \frac{1}{c} \leq V(y,F) \leq \frac{1}{c}(|F|^p + 1)$$

Homogenization of convex integral functionals

$$\mathcal{E}_\varepsilon(u) := \int_\Omega V(\tfrac{x}{\varepsilon}, \nabla u(x))\,dx$$

Suppose: $V(y,F)$ $\square$-periodic in y; convex & p-growth in F

Theorem: [Marcellini'77]. $\mathcal{E}_\varepsilon$ Γ-converges (in $L^p(\Omega)$) to

$$\mathcal{E}_0(u) := \int_\Omega V_{\text{hom}}^{(1)}(\nabla u(x))\,dx;$$

one-cell homogenization formula

$$V_{\text{hom}}^{(1)}(F) := \min_{\phi \in W^{1,p}_{\text{per}}(\square)} \fint_\square V(y, F + \nabla\phi(y))$$

Homogenization of convex integral functionals

$$\mathcal{E}_\varepsilon(u) := \int_\Omega V(\tfrac{x}{\varepsilon}, \nabla u(x))\,dx$$

Suppose: $V(y,F)$ $\square$-periodic in y; convex & p-growth in F

Theorem: [Marcellini'77]. $\mathcal{E}_\varepsilon$ Γ-converges (in $L^p(\Omega)$) to

$$\mathcal{E}_0(u) := \int_\Omega V_{\text{hom}}^{(1)}(\nabla u(x))\,dx;$$

one-cell homogenization formula

$$V_{\text{hom}}^{(1)}(F) := \min_{\phi \in W^{1,p}_{\text{per}}(\square)} \fint_\square V(y, F + \nabla\phi(y))$$

Quadratic-convex case: $Q_{\text{hom}}^{(1)}(F) = \fint_\square Q(y, F + \nabla\phi_F(y))$, where

$$-\nabla\cdot(\mathbb{L}(F + \nabla\phi_F)) = 0 \qquad \text{on } \square \text{ (with periodic BC)}$$

(periodic corrector equation)

Homogenization of non-convex integral functionals

$$\mathcal{E}_\varepsilon(u) := \int_\Omega W(\tfrac{x}{\varepsilon}, \nabla u(x))\, dx$$

Suppose: $W(y, F)$ $\square$-periodic in y; non-convex & p-growth in F.

Homogenization of non-convex integral functionals

$$\mathcal{E}_\varepsilon(u) := \int_\Omega W(\tfrac{x}{\varepsilon}, \nabla u(x))\,dx$$

Suppose: $W(y,F)$ $\square$-periodic in y; non-convex & p-growth in F.

Theorem: [Braides'85, Müller'87]. $\mathcal{E}_\varepsilon$ Γ-converges (in $L^p(\Omega)$) to

$$\mathcal{E}_0(u) := \int_\Omega W_{\text{hom}}(\nabla u(x))\,dx;$$

multi-cell homogenization formula

$$W_{\text{hom}}(F) := \inf_{k\in\mathbb{N}} \inf_{\phi\in W^{1,p}_{\text{per}}(k\square)} \fint_{k\square} W(y, F+\nabla\phi(y))$$

One-cell versus multi-cell formula

- **Convex case**:

$$V_{\rm hom}^{(1)}(F) := \min_{\phi \in W^{1,p}_{\rm per}(\Box)} \fint_{\Box} V(y, F + \nabla \phi(y)) = \fint_{\Box} V(y, F + \nabla \phi_F(y))$$

notion of corrector $\nabla \phi_F$ $\Rightarrow$ corrector based analysis, e.g.

$$\text{2scale expansion } \nabla u_\varepsilon \approx \nabla u_0 + \nabla \phi_{\nabla u_0}(\tfrac{\cdot}{\varepsilon})$$

- **Non-convex case**:

$$W_{\rm hom}(F) := \inf_{k \in \mathbb{N}} \inf_{\phi \in W^{1,p}_{\rm per}(k\Box)} \fint_{k\Box} W(y, F + \nabla \phi(y))$$

  - no corrector; evaluation of $W_{\rm hom}(F)$ difficult in practice.
  - Regularity of $W_{\rm hom}$? Behavior ∇u_ε?

One-cell versus multi-cell formula

- **Convex case**:

$$V_{\mathrm{hom}}^{(1)}(F) := \min_{\phi \in W^{1,p}_{\mathrm{per}}(\Box)} \fint_{\Box} V(y, F + \nabla\phi(y)) = \fint_{\Box} V(y, F + \nabla\phi_F(y))$$

notion of corrector $\nabla\phi_F \Rightarrow$ corrector based analysis, e.g.

2scale expansion $\nabla u_\varepsilon \approx \nabla u_0 + \nabla\phi_{\nabla u_0}(\frac{\cdot}{\varepsilon})$

- **Non-convex case**:

$$W_{\mathrm{hom}}(F) := \inf_{k \in \mathbb{N}} \inf_{\phi \in W^{1,p}_{\mathrm{per}}(k\Box)} \fint_{k\Box} W(y, F + \nabla\phi(y))$$

  - no corrector; evaluation of $W_{\mathrm{hom}}(F)$ difficult in practice.
  - Regularity of W_{hom}? Behavior ∇u_ε?

Can we have $W_{\mathrm{hom}}(F) = W_{\mathrm{hom}}^{(1)}(F)$?

One-cell versus multi-cell formula

☑ Impact of convexity [Müller '87]:

$$W(y,F) \text{ convex in } F \qquad \Rightarrow \qquad W_{\text{hom}}(F) = W_{\text{hom}}^{(1)}(F).$$

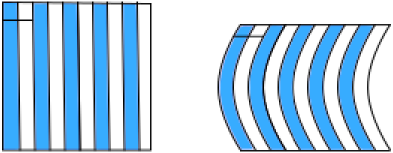
One-cell versus multi-cell formula

☑ Impact of convexity [Müller '87]:

$$W(y,F) \text{ convex in } F \qquad \Rightarrow \qquad W_{\text{hom}}(F) = W^{(1)}_{\text{hom}}(F).$$

☒ Buckling of microstructure [Müller '87]
Layered stiff/soft elastic two-phase composite

$$\Rightarrow \ \exists F(\text{compression}) \text{ s.t.} \qquad W_{\text{hom}}(F) < W^{(1)}_{\text{hom}}(F)$$



In fact: $\forall \delta > 0$ stiff/soft contrast can be chosen such that $W_{\text{hom}}(F) < W^{(1)}_{\text{hom}}(F)$ for some $\text{dist}(F, SO(d)) < \delta$

One-cell versus multi-cell formula

- ☒ [Barchiesi, Gloria '10]: $W_{\text{hom}}(F) < \mathcal{Q}W^{(1)}_{\text{hom}}(F)$
- ☑ Homogenization and linearization commute at Id

$$
\begin{array}{ccc}
\frac{1}{h^2}\int W(\frac{x}{\varepsilon}, \mathrm{Id}+h\nabla\varphi) & \xrightarrow{\text{lin}} & \int Q(\frac{x}{\varepsilon}, \nabla\varphi) \\
\big\downarrow \text{hom} & & \big\downarrow \text{hom} \\
\frac{1}{h^2}\int W_{\text{hom}}(\mathrm{Id}+h\nabla\varphi) & \xrightarrow{\text{lin}} & \int Q^{(1)}_{\text{hom}}(\nabla\varphi)
\end{array}
$$

[Müller, Neukamm '11, Gloria, Neukamm '12, Jesenko, Schmidt '14]

Main result:

Validity of single-cell formula (close to rotations)

Assumption **(A):** $W : \mathbb{R}^d \times \mathbb{R}^{d\times d} \to [0,+\infty]$ is

- $\Box$-periodic in first variable,

there exists $p \geq d$ and $\alpha > 0$ s.t. for a.e. $y \in \Box$ and every $F \in \mathbb{R}^{d\times d}$:

- $W(y, RF) = W(y,F)\ \forall R \in \mathrm{SO}(d)$ (**frame indifference**),
- $W(y, \mathrm{Id}) = \min W = 0$ (**reference configuration = natural state**)
- $W(y,F) \geq \alpha\,\mathrm{dist}^2(F, \mathrm{SO}(d))$ (**non-degeneracy**),
- $W(y,\cdot)$ is a C^3-function close to $\mathrm{SO}(d)$, (**regularity in** F)
- $\alpha|F|^p - \frac{1}{\alpha} \leq W(y,F)$ (**growth from below**)

Assumption **(A)** allows for physical growth

$$W(y,F) \to +\infty \quad \text{as} \quad \det(F) \to 0+\,.$$

Theorem: [with Neukamm]

Suppose **(A)** and regularity condition **(R)**.

Then $\exists \varrho > 0$ such that for all $F \in \mathbb{R}^{d\times d}$ with $\mathrm{dist}(F, \mathrm{SO}(d)) < \varrho$:

- (One-cell formula and corrector)

$$W_{\mathrm{hom}}(F) = W_{\mathrm{hom}}^{(1)}(F) = \fint_{\Box} W(y, F + \nabla\phi_F(y))\, dy$$

  for a corrector $\phi_F \in W^{1,p}_{\mathrm{per}}(\Box)$ (unique up to a constant)

- (Expansion of homogenization formula)

$$DW_{\mathrm{hom}}(F)[G] = \int_{\Box} DW(y, F + \nabla\phi_F(y))[G]\, dy$$

$$D^2 W_{\mathrm{hom}}(F)[G, G] = \inf_{\psi \in H^1_{\mathrm{per}}(\Box)} \int_{\Box} D^2 W(y, F + \nabla\phi_F)[G + \nabla\psi, G + \nabla\psi]\, dy$$

Ingredient I: Reduction to convex problem

Ingredient I: Reduction to convex problem

Lemma (matching convex lower-bound)

If **(A)**, then $\exists \beta, \mu, \delta > 0$ and $V : \mathbb{R}^d \times \mathbb{R}^{d\times d} \to \mathbb{R}$ s.t.

- $V(y,F)$ strongly β-convex in F, periodic in y
- $\beta|F|^2 - \frac{1}{\beta} \leq V(y,F) \leq \frac{1}{\beta}(1+|F|^2)$
- $V(y,\cdot) \in C^3(\mathbb{R}^{d\times d})$
- matching and lower-bound property

$$\begin{aligned} W(y,F) + \mu \det(F) &\geq V(y,F) \quad \text{for all } F \in \mathbb{R}^{d\times d} \\ W(y,F) + \mu \det(F) &= V(y,F) \quad \text{for } \operatorname{dist}(F, SO(d)) \leq \delta \end{aligned}$$

Variation of [Friesecke, Theil '02, Conti, Dolzmann, Kirchheim, Müller '06]

Fact: det is a Null-Lagrangian

$$\int_{\Box} \det(F + \nabla\phi) = \det(F) \quad \text{for all } \phi \in W^{1,d}_{\text{per}}(\Box)$$

Ingredient I: Reduction to convex problem

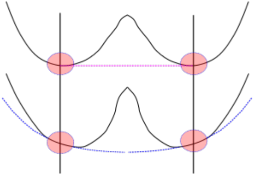
Lemma (matching convex lower-bound)

If **(A)**, then $\exists \beta, \mu, \delta > 0$ and $V : \mathbb{R}^d \times \mathbb{R}^{d\times d} \to \mathbb{R}$ s.t.

- $V(y, F)$ strongly β-convex in F, periodic in y
- $\beta|F|^2 - \frac{1}{\beta} \leq V(y,F) \leq \frac{1}{\beta}(1 + |F|^2)$
- $V(y, \cdot) \in C^3(\mathbb{R}^{d\times d})$
- matching and lower-bound property

$$
\begin{aligned}
W(y,F) + \mu \det(F) &\geq V(y,F) \quad \text{for all } F \in \mathbb{R}^{d\times d} \\
W(y,F) + \mu \det(F) &= V(y,F) \quad \text{for } \operatorname{dist}(F, SO(d)) \leq \delta
\end{aligned}
$$

Variation of [Friesecke, Theil '02, Conti, Dolzmann, Kirchheim, Müller '06]



$$D^2 W(y, \mathrm{Id})[G,G] \geq \alpha |\operatorname{sym} G|^2$$
$$D^2 \det(\mathrm{Id})[G,G] \geq |G|^2 - 2|\operatorname{sym} G|^2$$

Relate $W_{\hom}$ and $V_{\hom}$

- Poly-convex lower bound:

$$W_{\hom}(F) \geq V^{(1)}_{\hom}(F) - \mu \det(F) \quad \text{for all } F \in \mathbb{R}^{d\times d}$$

Proof: For all $k \in \mathbb{N}$ and $\psi \in W^{1,p}_{\mathrm{per}}(k\square)$:

$$\begin{aligned}\fint_{k\square} W(y, F+\nabla\psi)\,dy &\geq \fint_{k\square} V(y, F+\nabla\psi)\,dy - \mu \fint_{k\square} \det(F+\nabla\psi)\\ &= \fint_{k\square} V(y, F+\nabla\psi)\,dy - \mu \det(F)\\ &\geq \min_{\phi\in H^1_{\mathrm{per}}(\square)} \int_{\square} V(y, F+\nabla\phi)\,dy - \mu \det(F)\end{aligned}$$

Relate W_{hom} and V_{hom}

- Poly-convex lower bound:

$$W_{\mathrm{hom}}(F) \geq V^{(1)}_{\mathrm{hom}}(F) - \mu \det(F) \quad \text{for all } F \in \mathbb{R}^{d\times d}$$

- Exploit matching property:

$$\| \operatorname{dist}(F + \nabla\phi_F, SO(d))\|_{L^\infty(\Box)} < \delta \quad \Rightarrow \quad W_{\mathrm{hom}}(F) = W^{(1)}_{\mathrm{hom}}(F).$$

Proof:

$$\begin{aligned} W^{(1)}_{\mathrm{hom}}(F) &\leq \int_\Box W(y, F + \nabla\phi_F)\,dy \\ &= \int_\Box V(y, F + \nabla\phi_F)\,dy - \mu \int_\Box \det(F + \nabla\phi_F) \\ &= \min_{\phi \in H^1_{\mathrm{per}}(\Box)} \int_\Box V(y, F + \nabla\phi)\,dy - \mu \det(F) \end{aligned}$$

Relate W_{hom} and V_{hom}

- Poly-convex lower bound:

$$W_{\text{hom}}(F) \geq V_{\text{hom}}^{(1)}(F) - \mu \det(F) \quad \text{for all } F \in \mathbb{R}^{d\times d}$$

- Exploit matching property:

$$\| \operatorname{dist}(F + \nabla \phi_F, \mathrm{SO}(d))\|_{L^\infty(\Box)} < \delta \quad \Rightarrow \quad W_{\text{hom}}(F) = W_{\text{hom}}^{(1)}(F).$$

- Energy estimate:

$$\| \operatorname{dist}(F + \nabla \phi_F, \mathrm{SO}(d))\|_{L^2(\Box)} \lesssim \operatorname{dist}(F, \mathrm{SO}(d))$$

(not enough ☒).

Relate W_{hom} and V_{hom}

- Poly-convex lower bound:

$$W_{\text{hom}}(F) \geq V_{\text{hom}}^{(1)}(F) - \mu \det(F) \quad \text{for all } F \in \mathbb{R}^{d\times d}$$

- Exploit matching property:

$$\|\operatorname{dist}(F + \nabla\phi_F, \mathrm{SO}(d))\|_{L^\infty(\Box)} < \delta \quad \Rightarrow \quad W_{\text{hom}}(F) = W_{\text{hom}}^{(1)}(F).$$

- Energy estimate:

$$\|\operatorname{dist}(F + \nabla\phi_F, \mathrm{SO}(d))\|_{L^2(\Box)} \lesssim \operatorname{dist}(F, \mathrm{SO}(d))$$

(not enough ☒).

- Exploit **regularity condition (R)** to get Lipschitz estimate for ϕ_F:

$$\operatorname{dist}(F, \mathrm{SO}(d)) \ll 1 \;\Rightarrow\; \|\operatorname{dist}(F+\nabla\phi_F, \mathrm{SO}(d))\|_{L^\infty(\Box)} \lesssim \operatorname{dist}(F, \mathrm{SO}(d)).$$

Ingredient II: Lipschitz estimates

Ingredient II: Lipschitz estimates

Regularity condition (R) $\Rightarrow$ Lipschitz estimate (3 variants)

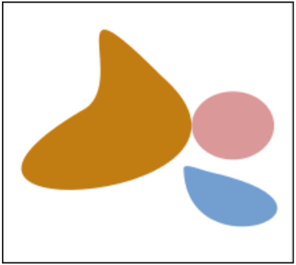
- **(R1)** (smooth): W is C^3 in neighbourhood of $\mathbb{R}^d \times \mathrm{SO}(d)$.
  Lipschitz estimate follows by implicit function theorem
- **(R2)** (laminate): $W(y, F) = W(y_d, F)$.
  Cell-problem reduces to ODE

Ingredient II: Lipschitz estimates

Regularity condition (R) $\Rightarrow$ Lipschitz estimate (3 variants)

- **(R1)** (smooth): W is C^3 in neighbourhood of $\mathbb{R}^d \times \mathrm{SO}(d)$.
- **(R2)** (laminate): $W(y, F) = W(y_d, F)$.
- **(R3)** (piecewise smooth composite)

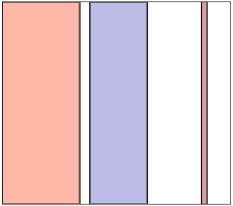
  disjoint (possibly touching) inclusions $D_1, \ldots, D_\ell$ with smooth boundary



[Li & Vogelius '00, Li & Nirenberg '03]:
Lipschitz estimates for linear elliptic systems
[Byun, Ryu & Wang '10, Byun & Kim '16 & 17]:
Gradient L^p, $p < \infty$ linear systems resp. scalar monotone equations

Geometric assumptions

- $\mathcal{D} := \{D_\ell\}_{\ell\in\mathbb{Z}}$ is called *laminate* if $\exists\ e \in \mathbb{R}^d$ and strictly monotone sequence $\{h_\ell\}_{\ell\in\mathbb{Z}}$ s.t. $D_\ell = \{x \in \mathbb{R}^d\ :\ h_\ell < x \cdot e < h_{\ell+1}\}\ \forall \ell \in \mathbb{Z}$.

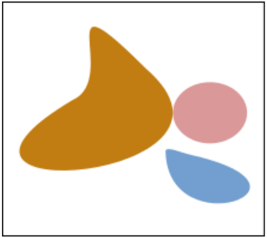


Interlude: Lipschitz estimates for elliptic systems
Geometric assumptions

- $\mathcal{D} := \{D_\ell\}_{\ell\in\mathbb{Z}}$ is called *laminate* if $\exists\ e \in \mathbb{R}^d$ and strictly monotone sequence $\{h_\ell\}_{\ell\in\mathbb{Z}}$ s.t. $D_\ell = \{x \in \mathbb{R}^d\ :\ h_\ell < x\cdot e < h_{\ell+1}\}\ \forall \ell \in \mathbb{Z}$.
- A sequence $\mathcal{D} := \{D_\ell\}_{\ell\in\mathbb{Z}}$ of mutually disjoint, open subsets of $\mathbb{R}^d$ such that $\mathbb{R}^d = \bigcup_{\ell\in\mathbb{Z}} \overline{D}_\ell$ is called *(E,s)-regular* (where $0 < s \leq 1$ and $E < \infty$), if for all $x \in \mathbb{R}^d$ there exists a laminate $\mathcal{D}_x$ s.t.

$$\sup_{0<r} r^{-s}\Big(|B_r|^{-1}\sum_{\ell\in\mathbb{Z}}\big|(D_\ell \triangle D'_{x\ell}) \cap B_r(x)\big|\Big)^{\frac{1}{2}} \leq E,$$

where $\triangle$ denote the symmetric difference.



Interlude: Lipschitz estimates for nonlinear elliptic systems
Assumptions on monotone operator

Definition: Given $\beta \in (0,1]$, we say $\mathbf{a} \in \mathcal{A}_\beta$ iff $\mathbf{a} : \mathbb{R}^{d\times d} \to \mathbb{R}^{d\times d}$ satisfies for all $F, G \in \mathbb{R}^{d\times d}$

$$\begin{aligned} \mathbf{a}(0) &= 0 \\ \beta|F-G|^2 &\leq \langle \mathbf{a}(F) - \mathbf{a}(G), F-G\rangle \\ \beta|\mathbf{a}(F) - \mathbf{a}(G)| &\leq |F-G| \\ \beta|D\mathbf{a}(F) - D\mathbf{a}(G)| &\leq \omega(|F-G|) \qquad \text{with } \omega(t) = \max\{t,1\} \end{aligned}$$

Fact: Suppose that W satisfies **(A)** then $\exists \beta \in (0,1]$ s.t. matching convex lower bound V satisfies $DV(x,\cdot) \in \mathcal{A}_\beta$ for a.e. $x \in \mathbb{R}^d$

Proposition: [with Neukamm]
Fix $\beta \in (0,1]$. Suppose that $\mathbf{a}$ is a (E,s)*-regular coefficient field of class* $\mathcal{A}_\beta$, i.e.

$$\mathbf{a}(y,F) = \sum_\ell \mathbf{a}_\ell(F)\mathbf{1}_{\mathcal{D}_\ell}(y),$$

where $\mathcal{D} = \{\mathcal{D}_\ell\}$ is (E,s)-regular and $\mathbf{a}_\ell \in \mathcal{A}_\beta$.
Given $q > d$, $\exists\ \overline{\kappa} > 0$ and $c \in [1,\infty)$ such that if $u \in H^1(B_1)$ and $f \in L^q(B_1)$ satisfy

$$-\operatorname{div} \mathbf{a}(x, \nabla u) = f \qquad \text{in } \mathscr{D}'(B_1),$$

and the smallness conditions

$$\|f\|_{L^q(B_1)} + \|\nabla u\|_{L^2(B_1)} \le \begin{cases} \infty & \text{if } d = 2 \\ \overline{\kappa} & \text{if } d \ge 3 \end{cases}.$$

Then,

$$\|\nabla u\|_{L^\infty(B_{\frac{1}{2}})} \le c\big(\|\nabla u\|_{L^2(B_1)} + \|f\|_{L^q(B_1)}\big).$$

Idea of the proof

Suppose

$$\operatorname{div} \mathbf{a}(x, \nabla u) = 0 \qquad \text{in } \mathscr{D}'(B_R).$$

Recall: *ε-Regularity statements:* (e.g. textbooks [Giaquinta], [Giusti])
Suppose $\mathbf{a}(x, \cdot) = \mathbf{a}(\cdot) \in \mathcal{A}_\beta$: For all $\alpha \in (0,1)$ there exists $\varepsilon > 0$ such that

$$E(\nabla u, B_R) := \left(\fint_{B_R} |\nabla u - \fint_{B_R} \nabla u|^2 \right)^{\frac{1}{2}} \leq \varepsilon$$
$$\Rightarrow E(\nabla u, B_r) \lesssim \left(\frac{r}{R} \right)^\alpha E(\nabla u, B_R)$$

'Proof':

(i) Differentiate eq.: $\operatorname{div}(\mathbb{L} \nabla \partial_i u) = 0$ with $\mathbb{L} := D\mathbf{a}(\nabla u)$

(ii) Let $w_i \in \partial_i u + H_0^1(\frac{1}{2}B)$ be such that

$$\operatorname{div} \overline{\mathbb{L}} \nabla w_i = 0 \quad \text{with} \quad \overline{\mathbb{L}} := D\mathbf{a}(\fint_B \nabla u)$$

(iii) Lipschitz estimates for w_i yield for $\tau \in (0,1]$:

$$E(\nabla u, \tau B) \lesssim \tau(1 + \tau^{-\frac{d}{2}} \omega(|E(\nabla u, B)|)^q) E(\nabla u, B)$$

Idea of the proof

Suppose

$$\operatorname{div}\mathbf{a}(x,\nabla u)=0\qquad\text{in }\mathscr{D}'(B_R).$$

(a) Let $\mathbf{a}$ be layered, i.e. $\mathbf{a}(x,\cdot)=\mathbf{a}(x_d,\cdot)$

- Previous argument & Lipschitz estimates for linear laminates [Chipot, Kinderlehrer, Caffarelli '85] yield

$$E(\nabla' u,B_R)\leq\varepsilon$$
$$\Rightarrow (E(\nabla' u,B_r)+E(J_d(u),B_r))\lesssim\left(\frac{r}{R}\right)^{\alpha}E(\nabla' u,B_R)$$

  where

$$\nabla' u:=(\partial_1 u,\ldots,\partial_{d-1}u)\qquad J_d(u):=\mathbf{a}(\cdot,\nabla u)e_d$$

- pointwise estimate

$$|\nabla w|\lesssim|\nabla' w|+|J_d(w)|\lesssim|\nabla w|$$

  yield Lipschitz-estimate.

(b) Treat (E,s)-regular coefficients as perturbations of layered coefficients.

□

Validity of

$$D^2 W_{\text{hom}}(F)[G,G] = \inf_{\psi \in H^1_{\text{per}}(\Box)} \int_{\Box} D^2 W(y, F + \nabla\phi_F)[G + \nabla\psi, G + \nabla\psi]\, dy$$

for all F with $\text{dist}(F, \text{SO}(d)) < \varrho$ and $G \in \mathbb{R}^{d\times d}$, follows by existence of Lipschitz corrector.

In [Geymonat, Müller, Triantafylidis '93] expansion is derived *assuming*

- single cell formula is valid and certain estimates of the corrector,
- or $W(y, F)$ is convex in F

From

$$D^2 W_{\text{hom}} = D^2 V^{(1)}_{\text{hom}} - \mu D^2 \det \quad \text{in } \{\text{dist}(\cdot, \text{SO}(d)) < \varrho\},$$

follows that W_{hom} is **strongly** rank-one convex in $\{\text{dist}(\cdot, \text{SO}(d)) < \varrho\}$, i.e.

$$D^2 W_{\text{hom}}(F)[a \otimes b, a \otimes b] \geq \beta |a|^2 |b|^2 \quad \text{for every } a, b \in \mathbb{R}^d.$$

Application I:
Quantitative two-scale expansion

Consider

$$\mathcal{I}_\varepsilon(u) := \int_\Omega W(\tfrac{x}{\varepsilon}, \nabla u) - f \cdot u \, dx,$$

$$\mathcal{I}_{\hom}(u) := \int_\Omega W_{\hom}(\nabla u) - f \cdot u \, dx,$$

subject to affine boundary condition $u(x) = Gx$ on $\partial\Omega$ (BC)

Theorem: [with Neukamm]

Let $r > d$. There exists $\bar{\varrho} > 0$. Suppose smallness of the data in form of

$$\Lambda(f, G) := \|f\|_{L^r(\Omega)} + \mathrm{dist}(G, SO(d)) < \bar{\varrho}.$$

(a) $\mathcal{I}_{\hom}$ admits a unique minimizer $u_0 \in W^{1,p}(\Omega)$ subject to (BC).

(b) Every minimizer $u_\varepsilon \in W^{1,p}(\Omega)$ of $\mathcal{I}_\varepsilon$ subject to (BC) satisfies

$$\|u_\varepsilon - u_0\|_{L^2(\Omega)} + \|u_\varepsilon - (u_0 + \varepsilon\phi_{\nabla u_0}(\tfrac{\cdot}{\varepsilon}))\|_{H^1(\Omega)} \lesssim \varepsilon^{\frac{1}{2}} \Lambda(f, G).$$

Consider

$$
\begin{aligned}
\mathcal{I}_\varepsilon(u) &:= \int_\Omega W(\tfrac{x}{\varepsilon}, \nabla u) - f \cdot u \, dx, \\
\mathcal{I}_{\text{hom}}(u) &:= \int_\Omega W_{\text{hom}}(\nabla u) - f \cdot u \, dx,
\end{aligned}
$$

subject to boundary condition $u = g$ on $\partial\Omega$ (BC)

Theorem: [with Neukamm]

Let $r > d$. There exists $\bar{\varrho} > 0$. Suppose smallness of the data in form of

$$\Lambda(f, g, g_0) := \|f\|_{L^r(\Omega)} + \|g - g_0\|_{W^{2,r}(\Omega)} + \|\operatorname{dist}(\nabla g_0, SO(d))\|_{L^\infty(\Omega)} < \bar{\varrho}.$$

where $g_0 \in W^{2,r}(\mathbb{R}^d)$ satisfies $-\operatorname{div} DW_{\text{hom}}(\nabla g_0) = 0$.

(a) $\mathcal{I}_{\text{hom}}$ admits a unique minimizer $u_0 \in W^{1,p}(\Omega)$ subject to (BC).

(b) For any $u_\varepsilon \in W^{1,p}(\Omega)$ satisfying (BC) have

$$
\begin{aligned}
&\|u_\varepsilon - u_0\|_{L^2(\Omega)} + \|u_\varepsilon - (u_0 + \varepsilon \phi_{\nabla u_0}(\tfrac{\cdot}{\varepsilon}))\|_{H^1(\Omega)} \\
&\qquad \lesssim \varepsilon^{\frac{1}{2}} \Lambda(f, g, g_0) + (\mathcal{I}_\varepsilon(u_\varepsilon) - \inf \mathcal{I}_\varepsilon)^{\frac{1}{2}} \\
&\qquad\quad + \varepsilon(1 + \|\nabla^2 g_0\|_{L^r(\Omega)}^{\frac{r}{r-d}})(\|\nabla^2 g_0\|_{L^r(\Omega)} + \Lambda(f, g, g_0)).
\end{aligned}
$$

Outline of the proof:

(I) Error estimate for matching convex lower bound

$$\mathcal{E}_\varepsilon(u) := \int_\Omega V(\tfrac{x}{\varepsilon}, \nabla u) - f \cdot u, \quad \mathcal{E}_{\mathrm{hom}}(u) := \int_\Omega V_{\mathrm{hom}}(\nabla u) - f \cdot u$$

- Minimizer of $\mathcal{E}_{\mathrm{hom}}$ in $g + H^1_0(\Omega)$ satisfies (via IFT)

$$u_0 \in W^{2,r}(\Omega), \quad \|\operatorname{dist}(\nabla u_0, \mathrm{SO}(d))\|_{L^\infty} \ll 1$$

$$\Rightarrow \qquad \mathcal{I}_{\mathrm{hom}}(u_0) = \min!.$$

- Estimate H^1-error of two-scale expansion

$$v_\varepsilon := u_0 + \varepsilon \eta_\varepsilon \phi_{\nabla u_0}(\tfrac{\cdot}{\varepsilon}) \in g + W^{1,r}_0(\Omega)$$

(adapt [Cardone, Zhikov, Pastukhova '06])

Outline of the proof:

(II) Lift estimate to non-convex problem

Assume: minimizer u_ε of $\mathcal{E}_\varepsilon$ satisfies $\|\operatorname{dist}(\nabla u_\varepsilon, \mathrm{SO}(d))\|_{L^\infty(\Omega)} < \delta$.

- u_ε is a unique minimizer of $\mathcal{I}_\varepsilon$
- For all $w \in W^{1,p}_g(\Omega)$ and two-scale expansion with cut-off v_ε have

$$
\begin{aligned}
\frac{1}{2}\|\nabla w - \nabla v_\varepsilon\|^2_{L^2(\Omega)} &\leq \|\nabla w - \nabla u_\varepsilon\|^2_{L^2(\Omega)} + \|\nabla v_\varepsilon - \nabla u_\varepsilon\|^2_{L^2(\Omega)} \\
&\lesssim \mathcal{E}_\varepsilon(w) - \mathcal{E}_\varepsilon(u_\varepsilon) + \|\nabla v_\varepsilon - \nabla u_\varepsilon\|^2_{L^2(\Omega)} \\
&\lesssim (\mathcal{I}_\varepsilon(w) - \inf \mathcal{I}_\varepsilon) + \varepsilon \Lambda(f,g)^2
\end{aligned}
$$

☒ Problem: In general no uniform Lipschitz estimate for u_ε.

Outline of the proof:

(II) Lift estimate to non-convex problem

Assume: two-scale expansion v_ε satisfies $\|\operatorname{dist}(\nabla v_\varepsilon, \mathrm{SO}(d))\|_{L^\infty(\Omega)} < \delta$.

- For all $w \in W^{1,p}_g(\Omega)$ have

$$\begin{aligned}\frac{1}{2}\|\nabla w - \nabla v_\varepsilon\|^2_{L^2(\Omega)} &\leq \|\nabla w - \nabla u_\varepsilon\|^2_{L^2(\Omega)} + \|\nabla v_\varepsilon - \nabla u_\varepsilon\|^2_{L^2(\Omega)} \\ &\lesssim \mathcal{E}_\varepsilon(w) - \mathcal{E}_\varepsilon(u_\varepsilon) + \|\nabla v_\varepsilon - \nabla u_\varepsilon\|^2_{L^2(\Omega)} \\ &\leq \mathcal{E}_\varepsilon(w) - \mathcal{E}_\varepsilon(v_\varepsilon) + \mathcal{E}_\varepsilon(v_\varepsilon) - \mathcal{E}_\varepsilon(u_\varepsilon) + \|\nabla v_\varepsilon - \nabla u_\varepsilon\|^2_{L^2(\Omega)} \\ &\lesssim (\mathcal{I}_\varepsilon(w) - \inf \mathcal{I}_\varepsilon) + \varepsilon\Lambda(f,g)^2\end{aligned}$$

☒ Problem: If $u_0 \notin W^{2,\infty}(\Omega)$ then $\operatorname{dist}(\nabla v_\varepsilon, \mathrm{SO}(d))$ not small,
$\nabla v_\varepsilon = \nabla u_0 + \eta_\varepsilon \nabla \phi_{\nabla u_0}(\tfrac{\cdot}{\varepsilon}) + \varepsilon \phi_{\nabla u_0} \otimes \nabla \eta_\varepsilon + \varepsilon \eta_\varepsilon D_F \phi_{\nabla u_0}[\nabla^2 u_0]$.

Outline of the proof:

(II) Lift estimate to non-convex problem

Consider modified two-scale expansion:

$\bar{v}_\varepsilon := u_0 + \varepsilon \eta_\varepsilon \phi_{(\nabla u_0)_\varepsilon}(\frac{\cdot}{\varepsilon})$

with $(\nabla u_0)_\varepsilon$ = Lipschitz-truncation of $\nabla u_0 \in W^{1,r}(\Omega)$.

- $\|\bar{v}_\varepsilon - v_\varepsilon\|_{H^1(\Omega)} \lesssim \varepsilon(1 + \|\nabla^2 g_0\|_{L^r(\Omega)}^{\frac{r}{r-d}})(\Lambda(f,g,g_0) + \|\nabla^2 g_0\|_{L^r(\Omega)})$
- For all $w \in W^{1,p}_g(\Omega)$ have

$$\frac{1}{2}\|\nabla w - \nabla v_\varepsilon\|_{L^2(\Omega)}^2 \leq (\mathcal{I}_\varepsilon(w) - \inf \mathcal{I}_\varepsilon) + \varepsilon \Lambda(f,g)^2 + \|\nabla v_\varepsilon - \nabla \bar{v}_\varepsilon\|_{L^2(\Omega)}^2$$

$\square$

Application II:
Uniform Lipschitz estimates for well-prepared & small data

Estimates for linear systems

- Lipschitz-estimate for harmonic functions: $\exists c = c(d) < \infty$

$$-\Delta u = 0 \quad \text{in } \mathscr{D}'(B) \quad \Rightarrow \quad \|\nabla u\|^2_{L^\infty(\frac{1}{2}B)} \leq c \fint_B |\nabla u|^2$$

Estimates for linear systems

- Lipschitz-estimate for harmonic functions: $\exists c = c(d) < \infty$

$$-\Delta u = 0 \quad \text{in } \mathscr{D}'(B) \quad \Rightarrow \quad \|\nabla u\|^2_{L^\infty(\frac{1}{2}B)} \leq c \fint_B |\nabla u|^2$$

- [Avellaneda, Lin '87] Suppose that $\mathbb{L} \in C^{0,\alpha}(\mathbb{R}^d, \mathbb{R}^{d^4})$ is periodic & uniformly elliptic, set $\mathbb{L}_\varepsilon := \mathbb{L}(\frac{\cdot}{\varepsilon})$. $\exists c < \infty$ such that for all $\varepsilon \in (0,1)$:

$$\operatorname{div}(\mathbb{L}_\varepsilon \nabla u) = 0 \quad \text{in } \mathscr{D}'(B) \quad \Rightarrow \quad \|\nabla u\|^2_{L^\infty(\frac{1}{2}B)} \leq c \fint_B |\nabla u|^2$$

[Li, Nirenberg '03] same estimate with piecewise $C^{0,\alpha}$ assumption

Estimates for linear systems

- Lipschitz-estimate for harmonic functions: $\exists c = c(d) < \infty$

$$-\Delta u = 0 \quad \text{in } \mathscr{D}'(B) \quad \Rightarrow \quad \|\nabla u\|^2_{L^\infty(\frac{1}{2}B)} \leq c \fint_B |\nabla u|^2$$

- [Avellaneda, Lin '87] Suppose that $\mathbb{L} \in C^{0,\alpha}(\mathbb{R}^d, \mathbb{R}^{d^4})$ is periodic & uniformly elliptic, set $\mathbb{L}_\varepsilon := \mathbb{L}(\frac{\cdot}{\varepsilon})$. $\exists c < \infty$ such that for all $\varepsilon \in (0,1)$:

$$\operatorname{div}(\mathbb{L}_\varepsilon \nabla u) = 0 \quad \text{in } \mathscr{D}'(B) \quad \Rightarrow \quad \|\nabla u\|^2_{L^\infty(\frac{1}{2}B)} \leq c \fint_B |\nabla u|^2$$

  [Li, Nirenberg '03] same estimate with piecewise $C^{0,\alpha}$ assumption

- Recent extension to **random** setting, e.g. [Armstrong, Smart '14], [Armstrong, Mourrat '16], [Gloria, Neukamm, Otto '14],..

Consider

$$\mathcal{I}_\varepsilon(u) \quad := \quad \int_\square W(\tfrac{x}{\varepsilon}, \nabla u) - f \cdot u \, dx,$$

subject to periodic boundary condition $u \in Gx + W^{1,p}_{\mathrm{per}}(\square)$ (pBC)

Theorem: [with Neukamm]
Let $q > d$ and $\varepsilon_n = \frac{1}{n}$, $n \in \mathbb{N}$. There exists $\bar{\varrho} > 0$. Suppose smallness of the data in form of

$$\Lambda(f, G) := \|f\|_{L^q(\square)} + \operatorname{dist}(G, SO(d)) < \bar{\varrho}.$$

(a) (Existence & uniqueness) $\mathcal{I}_{\varepsilon_n}$ admits a unique (up to a constant) minimizer $u_\varepsilon \in W^{1,p}(\square)$ subject to (pBC).

(b) (Lipschitz estimate & Euler-Lagrange equation) Every minimizer $u_{\varepsilon_n} \in W^{1,p}(\square)$ of $\mathcal{I}_{\varepsilon_n}$ subject to (pBC) satisfies

$$\|\operatorname{dist}(\nabla u_{\varepsilon_n}, SO(d))\|_{L^\infty(\square)} \lesssim \operatorname{dist}(G, SO(d)) + \|f\|_{L^q(\square)}$$

and

$$-\operatorname{div} DW(\tfrac{x}{\varepsilon_n}, \nabla u_{\varepsilon_n}) = f \qquad \text{in } \mathscr{D}'(\mathbb{R}^d)$$

Idea of proof

(a) Uniform Lipschitz estimate for monotone systems.
Suppose $\mathbf{a}$ periodic & (E,s)-regular coefficient field of class $\mathcal{A}_\beta$.
Then,

$$\operatorname{div} \mathbf{a}(x, \nabla u) = 0 \qquad \text{in } \mathscr{D}'(B)$$

and

$$\left(\fint_B |\nabla u|^2 \right)^{\frac{1}{2}} \leq \begin{cases} \infty & \text{if } d = 2, \\ \kappa & \text{if } d \geq 3, \end{cases}$$

imply

$$\|\nabla u\|_{L^\infty(\frac{1}{2}B)} \lesssim \left(\fint_B |\nabla u|^2 \right)^{\frac{1}{2}}$$

(b) Use (a) and the matching property to show that minimizer of the convex problem and non-convex problem coincide

Idea of proof (of (a)) (following [Avellaneda, Lin '87]).

Suppose

$$\operatorname{div} \mathbf{a}(x, \nabla u) = 0 \qquad \text{in } \mathscr{D}'(B_R) \text{ with } \quad 1 \ll R$$

$\exists\ \gamma = \gamma(\beta, d) \in (0,1)$ and $\kappa(\beta, d) > 0$ such that if

$$\widetilde{E}(\nabla u, B_R) := \inf_{\zeta} \left(\fint_{B_R} |\nabla u - (\zeta + \nabla \phi_\zeta)|^2 \right)^{\frac{1}{2}} \leq \begin{cases} +\infty & \text{if } d = 2, \\ \kappa & \text{if } d \geq 3 \end{cases}$$

Then,

- *Excess decay:* for all $\gamma' \in (0, \gamma)$

$$\widetilde{E}(\nabla u, B_r) \lesssim_{\gamma'} \left(\frac{r}{R}\right)^{\gamma'} \widetilde{E}(\nabla u, B_R) \qquad \text{for all } r \geq 1,$$

Idea of proof (of (a)) (following [Avellaneda, Lin '87]).

Suppose

$$\operatorname{div} \mathbf{a}(x, \nabla u) = 0 \qquad \text{in } \mathscr{D}'(B_R) \text{ with } \quad 1 \ll R$$

$\exists\, \gamma = \gamma(\beta, d) \in (0,1)$ and $\kappa(\beta, d) > 0$ such that if

$$\widetilde{E}(\nabla u, B_R) := \inf_{\zeta} \left(\fint_{B_R} |\nabla u - (\zeta + \nabla \phi_\zeta)|^2 \right)^{\frac{1}{2}} \leq \begin{cases} +\infty & \text{if } d = 2, \\ \kappa & \text{if } d \geq 3 \end{cases}$$

Then,

- *Excess decay:* for all $\gamma' \in (0, \gamma)$

$$\widetilde{E}(\nabla u, B_r) \lesssim_{\gamma'} \left(\frac{r}{R}\right)^{\gamma'} \widetilde{E}(\nabla u, B_R) \qquad \text{for all } r \geq 1,$$

  Proof: Compare u with suitable $\mathbf{a}_{\text{hom}}$-harmonic function $\Rightarrow \exists q > 0$ s.t. for all $\tau \in (0, 1]$

$$\widetilde{E}(\nabla u, B_{\tau R}) \lesssim \left(\tau^\gamma + \tfrac{1}{R^q} \tau^{-\frac{d}{2}} \right) \widetilde{E}(\nabla u, B_R)$$

  where

  - $\tau^\gamma \sim$ regularity for homogenized problem
  - $R^{-q} \sim$ homogenization error

Idea of proof (of (a)) (following [Avellaneda, Lin '87]).

Suppose

$$\operatorname{div} \mathbf{a}(x, \nabla u) = 0 \qquad \text{in } \mathscr{D}'(B_R) \text{ with } \quad 1 \ll R$$

$\exists\ \gamma = \gamma(\beta, d) \in (0,1)$ and $\kappa(\beta, d) > 0$ such that if

$$\widetilde{E}(\nabla u, B_R) := \inf_{\zeta} \left(\fint_{B_R} |\nabla u - (\zeta + \nabla \phi_\zeta)|^2 \right)^{\frac{1}{2}} \leq \begin{cases} +\infty & \text{if } d = 2, \\ \kappa & \text{if } d \geq 3 \end{cases}$$

Then,

- *Excess decay:* for all $\gamma' \in (0, \gamma)$

$$\widetilde{E}(\nabla u, B_r) \lesssim_{\gamma'} \left(\frac{r}{R}\right)^{\gamma'} \widetilde{E}(\nabla u, B_R) \qquad \text{for all } r \geq 1,$$

- *Large scale Lipschitz estimate*:

$$\fint_{B_1} |\nabla u|^2 \lesssim \fint_{B_R} |\nabla u|^2.$$

Idea of proof (of (a)) (following [Avellaneda, Lin '87]).

Suppose

$$\operatorname{div} \mathbf{a}(x, \nabla u) = 0 \qquad \text{in } \mathscr{D}'(B_R) \text{ with } \quad 1 \ll R$$

$\exists\, \gamma = \gamma(\beta, d) \in (0,1)$ and $\kappa(\beta, d) > 0$ such that if

$$\widetilde{E}(\nabla u, B_R) := \inf_{\zeta} \left(\fint_{B_R} |\nabla u - (\zeta + \nabla \phi_\zeta)|^2 \right)^{\frac{1}{2}} \leq \begin{cases} +\infty & \text{if } d = 2, \\ \kappa & \text{if } d \geq 3 \end{cases}$$

Then,

- *Excess decay:* for all $\gamma' \in (0, \gamma)$

$$\widetilde{E}(\nabla u, B_r) \lesssim_{\gamma'} \left(\frac{r}{R}\right)^{\gamma'} \widetilde{E}(\nabla u, B_R) \qquad \text{for all } r \geq 1,$$

- *Large scale Lipschitz estimate*:

$$\fint_{B_1} |\nabla u|^2 \lesssim \fint_{B_R} |\nabla u|^2.$$

- *Lipschitz estimate*: (Here use that $\mathbf{a}$ is (E, s)-regular)

$$\|\nabla u\|_{L^\infty(B_1)} \lesssim \fint_{B_R} |\nabla u|^2.$$

□

Summary:

- One-cell formula for deformation close to $SO(d)$
- Uniform Lipschitz estimate for small data
- Estimate on homogenization error for small data

Outlook:

- Homogenization/linearization in neighborhood of rotations
- Stochastic homogenization

Summary:

- One-cell formula for deformation close to $SO(d)$
- Uniform Lipschitz estimate for small data
- Estimate on homogenization error for small data

Outlook:

- Homogenization/linearization in neighborhood of rotations
- Stochastic homogenization

Thank you for your attention!

More informations,

S. Neukamm, M. S., Quantitative homogenization in non-linear elasticity for small loads, *Archive for Rational Mechanics and Analysis* (online first) *arXiv:1703.07947* (one cell formula / error estimate for smooth / layered coefficients)

- For $0 < \mu \ll 1$ set $\overline{W} := W + \mu \det$. $\exists\, \delta = \delta(d, \alpha, \mu)$, $C = C(d)$ s.t.
$$\overline{W}(F) - \overline{W}(F_0) - D\overline{W}(F_0)[F - F_0] \geq \tfrac{\mu}{C}|F - F_0|^2$$
for all $F, F_0 \in \mathbb{R}^{d\times d}$ with $\mathrm{dist}(F_0, \mathrm{SO}(d)) < \delta$
- Set $\lambda = \frac{\mu}{2C}$ and
$$\overline{V}(F) := \sup_{F_0 \in U_\delta} Q_{F_0}(F),$$
where
$$\overline{Q}_{F_0}(F) := \overline{W}(F_0) + D\overline{W}(F_0)[F - F_0] + \lambda|F - F_0|^2.$$
Then:
  - $\overline{V}$ strongly convex,
  - $\overline{V} \leq \overline{W}$, and $\overline{V} = \overline{W}$ on $\{\mathrm{dist}(F, \mathrm{SO}(d)) < \delta\}$.
- Regularization via smoothing and gluing
(carefull gluing is non-convex, smoothing increases function)