Recent Advances in Homogenisation Theory

Department of Mathematics
Durham University

18^{th} - 19^{th} June 2018

Welcome

It is my greatest pleasure to welcome everyone to the first Recent Advances in Homogenisation theory conference here at Durham. I would like to take the opportunity to thank you all for travelling to Durham and participating in this workshop.

I hope you enjoy the event and the opportunity to interact with the wider UK-based applied analysis community.

Shane Cooper

Programme

Monday

09:15- 09:45	Coffee and registration	
09:45-10:00	Welcome	
10:00-11:00	Valery Smyshlyaev	Two-scale homogenisation of high-contrast PDE systems
11:00-12:00	Michel Bellieud	Capacitary problems in fibered structures
12:00 - 13:30	Lunch	
13:30 - 14:30	Anton Savostianov	Homogenisation with error estimates of attractors for damped semi-linear anisotropic wave equations
14:30 - 15:00	Coffee	
15:00 - 16:00	Mikhail Cherdantsev	Stochastic homogenisation of high-contrast media
16:00 - 17:00	Igor Velčić	Sharp operator-norm asymptotics for linearised elastic plates with rapidly oscillating periodic properties
17:00 - 19:00	Discussion / free time	
19:00	Conference dinner	

Tuesday

09:30-10:30	Ilia Kamotski	Operator estimates for Schrodinger type equations with oscillatory coefficients
10:30 - 11:00	Coffee	
11:00 - 12:00	Harsha Hutridurga	Averaging and initial layer analysis in passive transport
12:00 - 13:30	Lunch	
13:30 - 14:30	Marcus Waurick	Nonlocal H-convergence
14:30 - 15:00	Coffee	
15:00 - 16:00	Mathias Schäffner	Quantitative homogenization in nonlinear elasticity for small loads
16:00 - 17:00	Georgy Kitavtsev	On applications of the implicit function theorem to singular perturbed elliptic PDEs
17:00	Close	

Abstracts

Two-scale homogenisation of high-contrast PDE systems

Monday
10:00am

Valery Smyshlyaev
University College London

There has been considerable recent interest in composite materials whose macroscopic physical properties can be very different from those of conventional materials, often due to effects of the so-called "micro-resonances". Mathematically this leads to studying high-contrast homogenisation of problems with a "critically scaled high contrast, where the resulting two-scale asymptotic behaviour appears to display a number of interesting effects. Mathematical analysis of these problems requires development of "two-scale" versions of operator and spectral convergences, of compactness, etc. We will review some background, as well as some more recent developments and applications. One is two-scale analysis of general "partially-degenerating" periodic PDE problems [1], where strong two-scale resolvent convergence appears to hold under a rather generic decomposition assumption, implying in particular (two-scale) convergence of semigroups with applications to a wide class of micro-resonant dynamic problems. We also discuss examples where the key assumption is not satisfied, however the conclusions still hold via to a two-scale homogenisation with respect to measures due to V.V. Zhikov.

A substantial additional effort is required for establishing not only the convergence but also its rate i.e. error bounds. In [2], we establish such error bounds for eigenvalue and eigenmodes due to a localized defect in a high-contrast periodic medium. Finally, time permitting, we will briefly review a most recent generic approach for establishing operator error bounds for high-contrast infinite periodic problems.

[1] I.V. Kamotski, V.P. Smyshlyaev, Two-scale homogenization for a general class of high contrast PDE systems with periodic coefficients, to appear in *Applicable Analysis* (2018), published online February 2018. Available also as https://arxiv.org/pdf/1309.4579v2.pdf

[2] I. V. Kamotski, V. P. Smyshlyaev, Localised modes due to defects in high contrast periodic media via two-scale homogenization, to appear in *J. Math. Sci (NY)* (2018). Available also as https://arxiv.org/pdf/1801.03372v1.pdf

---

Capacitary problems in fibered structures

Monday 11:00

Michel Bellieud

Université de Montpellier 2

We analyze an elastic or visco-plastic composite wherein stiff fibers of tiny cross sections are embedded in a softer matrix. The relative rotation and averaged displacement of the fibers with respect to the matrix generates a concentration of energy around the fibers characterized in terms of a variant of the notion of capacity introduced in [1]. This contribution interacts with a combination of torsional, bending, and stretching energy stored within the fibers.

[1] M. B.: A notion of capacity related to linear elasticity. Applications to homogenization. Arch. Ration. Mech. Anal., vol. 203, no1, (2012), pp. 137–187.

Homogenisation with error estimates of attractors for damped semi-linear anisotropic wave equations

Monday 13:30

Anton Savostianov

Durham University

Homogenisation of global $\mathcal{A}^\varepsilon$ and exponential $\mathcal{M}^\varepsilon$ attractors for the damped semi-linear anisotropic wave equation $\partial_t^2 u^\varepsilon + \gamma \partial_t u^\varepsilon - \operatorname{div}\left(a\left(\frac{x}{\varepsilon}\right)\nabla u^\varepsilon\right) + f(u^\varepsilon) = g$, on a bounded domain $\Omega \subset \mathbb{R}^3$, is performed. Order-sharp estimates between trajectories $u^\varepsilon(t)$ and their homogenised trajectories $u^0(t)$ are established. These estimates are given in terms of the operator-norm difference between resolvents of the elliptic operator $\operatorname{div}\left(a\left(\frac{x}{\varepsilon}\right)\nabla\right)$ and its homogenised limit $\operatorname{div}\left(a^h\nabla\right)$. Consequently, norm-resolvent estimates on the Hausdorff distance between the anisotropic attractors and their homogenised counter-parts $\mathcal{A}^0$ and $\mathcal{M}^0$ are established. These results imply error estimates of the form $\operatorname{dist}_X(\mathcal{A}^\varepsilon, \mathcal{A}^0) \le C\varepsilon^\varkappa$ and $\operatorname{dist}^s_X(\mathcal{M}^\varepsilon, \mathcal{M}^0) \le C\varepsilon^\varkappa$ in the spaces $X = L^2(\Omega) \times H^{-1}(\Omega)$ and $X = (C^\beta(\overline{\Omega}))^2$. In the natural energy space $\mathcal{E} := H^1_0(\Omega) \times L^2(\Omega)$, error estimates $\operatorname{dist}_{\mathcal{E}}(\mathcal{A}^\varepsilon, \mathrm{T}_\varepsilon\mathcal{A}^0) \le C\sqrt{\varepsilon^\varkappa}$ and $\operatorname{dist}^s_{\mathcal{E}}(\mathcal{M}^\varepsilon, \mathrm{T}_\varepsilon\mathcal{M}^0) \le C\sqrt{\varepsilon^\varkappa}$ are established where T_ε is first-order correction for the homogenised attractors suggested by asymptotic expansions. Our results are applied to Dirchlet, Neumann and periodic boundary conditions.

Stochastic homogenisation of high-contrast media

Monday 15:00

Mikhail Cherdantsev

Cardiff University

Using a suitable stochastic version of the compactness argument of V. V. Zhikov, we develop a probabilistic framework for the analysis of heterogeneous media with high contrast. We show that an appropriately defined multiscale limit of the field in the original medium satisfies a system of equations corresponding to the coupled "macroscopic" and "microscopic" components of the field, giving rise to an analogue of the "Zhikov function", which represents the effective dispersion of the medium. We demonstrate that, under some lenient conditions within the new framework, the spectra of the original problems converge to the spectrum of their homogenisation limit.

Sharp operator-norm asymptotics for linearised elastic plates with rapidly oscillating periodic properties

Monday
16:00

Igor Velčić
University of Zagreb

In this talk we analyse a system of partial differential equations describing the behaviour of an elastic plate with periodic moduli in the two planar directions. We assume that the displacement gradients of the points of the plate are small enough for the equations of linearised elasticity to be a suitable approximation of the material response. Following the application of an appropriate version of the Floquet transform, we analyse the operator-norm resolvent behaviour of the operators in each fibre of the resulting direct integral, as the period and the plate thickness go to zero simultaneously. The convergence estimates we obtain are uniform with respect to both the Floquet parameter and the plate thickness, which yields order-sharp resolvent estimates for the convergence of the original plate problems as the plate thickness goes to zero.

We use the approach of Cherednichenko, Cooper (ARMA 219, 1061-1086 (2016)) where they analyzed high-contrast elliptic equation. This is a joint work with Kirill Cherednichenko (University of Bath).

Operator estimates for Schrodinger type equations with oscillatory coefficients

Tuesday
09:30

Ilia Kamotski
University College London

Averaging and initial layer analysis in passive transport

Tuesday
11:00

Harsha Hutridurga
Imperial College London

Passive transport models are equations of advection-diffusion type. In most of the applications involving passive transport, the advective fields are of greater magnitude compared to molecular diffusion. This talk attempts to present a novel theory developed by myself, Thomas Holding (Imperial) and Jeffrey Rauch (Michigan) to address these strong advection problems. Loosely speaking, our strategy is to recast the advection-diffusion equation in moving coordinates dictated by the flow associated with the advective field. Crucial to our analysis is the introduction of a fast time variable and the introduction of some new notions of weak convergence along flows in L^p spaces. We also use ideas from the theory of "homogenization structures" developed by Gabriel Nguetseng.

Our asymptotic results show the following dichotomy:

- If the Jacobian matrix associated with the flow satisfies certain structural conditions (loosely speaking, boundedness in the fast time variable) then the strong advection limit is a non-degenerate diffusion when seen along flows.

- On the other hand, when the Jacobian matrix associated with the flow fails to satisfy the aforementioned structural conditions, then the strong advection limit is a parabolic problem with a constraint. Here we show the appearance of an initial layer where there is an enhanced dissipation along flows.

Our results have close links to

- the Freidlin-Wentzell theory on perturbations of dynamical systems.
- the theory of Relaxation enhancing Lipschitz flows.

This talk will illustrate the theoretical results via various interesting examples. We address some well-known advective fields such as the Euclidean motions, the Taylor-Green cellular flows, the cats eye flows and some special class of the Arnold-Beltrami-Childress (ABC) flows. We will also comment on certain examples of hyperbolic or Anosov flows.

Some of the results to be presented in this talk can be found in the following publication:

T. Holding, H. Hutridurga, J. Rauch. Convergence along mean flows, SIAM J Math. Anal., Volume 49, Issue 1, pp. 222271 (2017).

Nonlocal H-convergence

Tuesday
13:30

Marcus Waurick
University of Strathclyde

In the talk I will introduce a notion of convergence -- so-called nonlocal H-convergence -- of possibly nonlocal coefficients in elliptic divergence form problems. It will turn out that the thus induced topology cannot be compared with the weak operator topology but is strictly weaker than both the strong operator topology and operator norm topology. In the case of multiplication operators the introduced notion coincides with H-convergence as introduced by Tartar and Murat. We will provide an adapted compactness statement for nonlocal H-convergence and a div-curl type characterisation for this convergence. An integral part of the definition is the notion of exact operator complexes in Hilbert spaces. With this it is possible to directly apply the concept to differential equations on manifolds and/or to problems in electromagnetic theory.

The results can be found in arXiv:1804.02026.

Quantitative homogenization in nonlinear elasticity for small loads

Tuesday 15:00

Mathias Schäffner
TU Dresden

We consider a nonlinear elastic composite with a periodic micro-structure described by the non-convex integral functional

$$I_\varepsilon(u) = \int_\Omega W\left(\frac{x}{\varepsilon}, \nabla u(x)\right) - f(x) \cdot u(x)\, dx.$$

As it is well-known, under suitable growth conditions, I_ε Γ-converges to a functional with a homogenized energy density $W_{\text{hom}}(F)$, which is given by an *infinite-cell formula*. Under appropriate assumptions on W (e.g. frame indifference, minimality at identity, non-degeneracy and smoothness in a neighborhood close to the set of rotations) and on the microstructure (e.g. matrix material and a finite number of smooth but possibly touching inclusions), we show that in a neighbourhood of rotations the homogenized stored energy function W_{hom} is characterized by a *single-cell homogenization formula*. For this, we combine the construction of a matching convex lower bound and Lipschitz-estimates for sufficiently flat solutions of nonlinear elliptic systems with piecewise smooth coefficients that depend on the shape and the size of the surfaces of discontinuity but are independent of the distance between these surfaces.

Moreover, for small and well-prepared data, we establish existence and uniform Lipschitz estimates for minimizers of I_ε and establish a quantitative two-scale expansion. This is joint work with Stefan Neukamm (TU Dresden) and parts of the results can be found in

- S. Neukamm and M. Schäffner, Quantitative homogenization in nonlinear elasticity for small loads, *Arch. Ration. Mech. Anal.*, (2018) (online first), *arXiv:1703.07947*

On applications of the implicit function theorem to singular perturbed elliptic PDEs

Tuesday 16:00

Georgy Kitavtsev
Oxford University

In this talk I will present two applications of a modification of the implicit function theorem developed in Magnus [2006] and Recke and Omelchenko [2008], which has minimal assumptions concerning continuity with respect to the control parameter, to singular perturbed elliptic PDEs. The first application concerns the spectral problem for the fourth-order linear operator arising in modelling of thin liquid films with essential degeneracy as the film thickness goes to zero. In the second application we will demonstrate existence of axially symmetric minimisers to the Q-tensor harmonic map problem arising in the Landau-de Gennes theory of nematic liquid crystals under singularly perturbed anchoring boundary conditions.
