Stochastic homogenisation of high-contrast media

M. Cherdantsev, Cardiff University

(joint work with K. Cherednichenko, University of Bath,

and I. Velčić, University of Zagreb)

Recent advances in homogenisation theory

Durham

18 June 2018

Classical periodic homogenisation

Maxwell, Rayleigh, Einstein, Bakhvalov, Panasenko, Bensoussan, Lions, Papanicolaou, Marchenko, Khruslov, Zhikov, Kozlov, Oleinik, Shamaev, Yosifian, Sanchez-Palencia

Operators with periodic coefficients

$$\mathcal{A}_\varepsilon := -\nabla \cdot a\left(x/\varepsilon\right)\nabla, \quad \varepsilon \to 0$$

$$a(y) \text{ is } Q\text{-periodic}, \quad \xi \cdot a(y)\xi \geq C|\xi|^2, \ C > 0$$

$$\mathcal{A}_\varepsilon u_\varepsilon = f$$

Asymptotics of the solution

$$u_\varepsilon(x) \approx u_0(x) + \varepsilon u_1(x, x/\varepsilon) + \ldots$$

$$u_1(x,y) = \frac{\partial u_0(x)}{\partial x_k} N_k(y)$$

where N_k is periodic and solve the cell problem

$$\nabla_y \cdot a(y)(\mathbf{e}_k + \nabla_y N_k(y)) = \frac{\partial}{\partial y_i} a_{ij}(y)\big(\delta_{jk} + \frac{\partial}{\partial y_j} N_k(y)\big) = 0$$

Homogenised operator

$$\mathcal{A}_{\text{hom}} u_0 = -\nabla \cdot a^{\text{hom}} \nabla u_0 = f$$

$$a_{ik}^{\text{hom}} = \int_Q a_{ij}(y)\big(\delta_{jk} + \frac{\partial}{\partial y_j} N_k(y)\big)\, dy$$

Classical periodic homogenisation

Operators with periodic coefficients

$$\mathcal{A}_\varepsilon := -\nabla \cdot a\left(x/\varepsilon\right)\nabla, \quad \varepsilon \to 0$$

$$\mathcal{A}_\varepsilon u_\varepsilon = f$$

Asymptotics of the solution

$$u_\varepsilon(x) \approx u_0(x) + \varepsilon u_1(x, x/\varepsilon) + \dots$$

$$u_1(x,y) = \frac{\partial u_0(x)}{\partial x_k} N_k(y)$$

where N_k is periodic and solve the cell problem

$$\nabla_y \cdot a(y)\nabla_y(\mathbf{e}_k + N_k(y)) = \frac{\partial}{\partial y_i} a_{ij}(y)\big(\delta_{jk} + \frac{\partial}{\partial y_j} N_k(y)\big) = 0$$

Homogenised operator

$$\mathcal{A}_{\mathrm{hom}} u_0 = -\nabla \cdot a^{\mathrm{hom}} \nabla u_0 = f$$

$$a_{ik}^{\mathrm{hom}} = \int_Q a_{ij}(y)\big(\delta_{jk} + \frac{\partial}{\partial y_j} N_k(y)\big)\, dy$$

Equivalently,

$$\xi \cdot a^{\mathrm{hom}} \xi = \inf_{\psi \in H^1_{\mathrm{per}}(Q)} \int_Q (\xi + \nabla_y \psi) \cdot a(y) (\xi + \nabla_y \psi)\, dy$$

Probability framework
Let $(\Omega, \mathcal{F}, P)$ be a complete probability space.

Definition

A family $(T_x)_{x \in \mathbf{R}^n}$ of measurable bijective mappings $T_x : \Omega \to \Omega$ on a probability space $(\Omega, \mathcal{F}, P)$ is called a dynamical system on Ω with respect to P if:

1. $T_x \circ T_y = T_{x+y} \quad \forall\, x, y \in \mathbf{R}^n$;
2. $P(T_x F) = P(F) \quad \forall x \in \mathbf{R}^n,\ F \in \mathcal{F}$;
3. $\mathcal{T} : \mathbf{R}^n \times \Omega \to \Omega, \quad (x, \omega) \to T_x(\omega)$ is measurable (for the standard σ-algebra on the product space, where on $\mathbf{R}^n$ we take the Lebesgue σ-algebra).

Definition

A dynamical system is called ergodic if one of the following equivalent conditions is fulfilled:

1. f measurable,
   $f(\omega) = f(T_x\omega)\ \forall x \in \mathbf{R}^n,\ \text{a.e. } \omega \in \Omega \implies f(\omega) \text{ is constant } P\text{-a.e. } \omega \in \Omega.$
2. $P\big((T_x B \cup B) \backslash (T_x B \cap B)\big) = 0\ \forall x \in \mathbf{R}^n \implies P(B) \in \{0, 1\}.$
3. $T_x B = B \quad \forall x \in \mathbf{R}^n \implies P(B) \in \{0, 1\}$

Operators with stochastic coefficients
Let $a \in L^\infty(\Omega)$, $\xi \cdot a(\omega)\xi \geq C|\xi|^2$, $C > 0$.
We write $a(y, \omega) := a(T_y \omega)$, defining the realisation of random coefficients.

$$\mathcal{A}_\varepsilon := -\nabla \cdot a\left(x/\varepsilon, \omega\right) \nabla, \quad \varepsilon \to 0$$

$$\mathcal{A}_\varepsilon u_\varepsilon = f$$

Probabilistic gradient

$$D_i f(\omega) := \lim_{x_i \to 0} \frac{f(T_{x_i}\omega) - f(\omega)}{x_i}, \qquad f \in L^2(\Omega),$$

D_i is the infinitesimal generator of the unitary group U_{x_i}

$$\nabla_\omega := (D_1, \dots, D_n)^{\mathrm{T}}, \qquad T_x \nabla_\omega f(\omega) = \nabla f(x, \omega) = \nabla(T_x f(\omega))$$

The "cell" problem

$$\nabla_\omega \cdot a(\omega)(\mathbf{e}_k + \nabla_\omega N_k(\omega)) = D_i a_{ij}(\omega)\big(\delta_{jk} + D_j N_k(\omega)\big) = 0$$

Homogenised operator

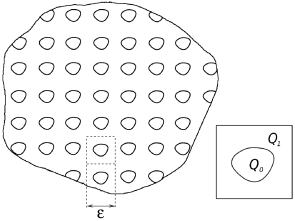
$$\mathcal{A}_{\mathrm{hom}} u_0 = -\nabla \cdot a^{\mathrm{hom}} \nabla u_0 = f$$

$$\xi \cdot a^{\mathrm{hom}} \xi = \inf_{\psi \in H^1(\Omega)} \int_\Omega (\xi + \nabla_\omega \psi) \cdot a(\omega)(\xi + \nabla_\omega \psi)\, dP$$

Operators with high-contrast periodic coefficients

$$\mathcal{A}_\varepsilon := -\nabla \cdot a(x,\varepsilon)\nabla$$

$$a(x,\varepsilon) := \begin{cases} \varepsilon^2, & x \in S_0^\varepsilon \\ a_1, & x \in S_1^\varepsilon \end{cases}$$



Asymptotics of the solution

$$u_\varepsilon(x) \approx u(x, x/\varepsilon) + \varepsilon u_1(x, x/\varepsilon) + \dots$$

$$u(x,y) = u_0(x) + v(x,y) \in H^1(\mathbb{R}^n) + L^2(\mathbb{R}^n; H_0^1(Q_0))$$

Homogenised operator is two-scale

$$\mathcal{A}_{\text{hom}} u(x,y) = f(x,y)$$

$$\begin{cases} -\nabla \cdot a_1^{\text{hom}} \nabla u_0(x) = \langle f(x,y)\rangle_y, & x \in \mathbb{R}^n, \\ -\Delta_y v = f(x,y), & y \in Q_0. \end{cases}$$

$$\xi \cdot a_1^{\text{hom}} \xi = \inf_{\psi \in H^1_{\text{per}}(Q_1)} \int_{Q_1} (\xi + \nabla_y \psi) \cdot a_1(y)(\xi + \nabla_y \psi)\, dy$$

Band-gap spectrum in high-contrast periodic homogenisation

$$-\nabla \cdot a_1^{\text{hom}} \nabla u_0 = \lambda \left(u_0 + \langle v \rangle \right),$$
$$-\Delta_y v = \lambda \left(u_0 + v \right).$$

$$v(x,y) = \lambda u_0(x) b(y)$$
$$-\Delta_y b = 1 + \lambda b$$

Decoupling:

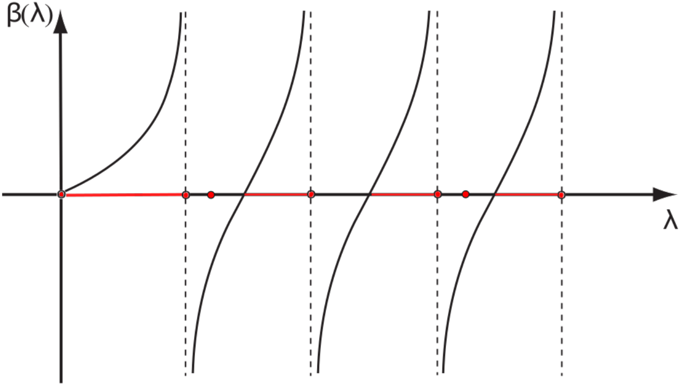
$$-\nabla \cdot a_1^{\text{hom}} \nabla u_0 = \beta(\lambda) u_0$$

$$\beta(\lambda) := \lambda + \lambda^2 \sum_{j=1}^{\infty} \frac{\langle \varphi_j \rangle^2}{\nu_j - \lambda}.$$

$$\sigma(\mathcal{A}_{\text{hom}}) = \{\lambda : \beta(\lambda) \geq 0\} \cup \{\nu_1', \nu_2', \dots\}$$

The spectrum of $\mathcal{A}_\varepsilon$ has gaps for sufficiently small ε (Zhikov (2000, 2004)).

Zhikov's β-function



Operators with stochastic high-contrast coefficients

$$\mathcal{A}_\varepsilon := -\nabla \cdot a\,(x,\varepsilon,\omega)\,\nabla$$
$$a(x,\varepsilon,\omega) := \begin{cases} \varepsilon^2, & x \in S_0^\varepsilon(\omega) \\ a_1, & x \in S_1^\varepsilon(\omega) \end{cases}$$

$S_0^\varepsilon(\omega)$ is a set of randomly distributed inclusions (balls, for example).
Let $\omega \in \Omega$, $\lambda < 0$, $f^\varepsilon \in L^2(S)$, $u^\varepsilon \in H_0^1(S)$

$$-\nabla \cdot a(x,\varepsilon,\omega)\nabla u^\varepsilon - \lambda u^\varepsilon = f^\varepsilon, \quad x \in S.$$

Theorem

Let $f^\varepsilon \overset{2}{\rightharpoonup} f \in L^2(S\times\Omega)$. Then for a.e. $\omega \in \Omega$ one has $u^\varepsilon \overset{2}{\rightharpoonup} u(x,\omega) = u_0(x) + u_1(x,\omega)$, where $u_0 \in H_0^1(S)$, $u_1 \in L^2(S, H_0^1(\mathcal{O}))$ satisfy

$$-\nabla \cdot a_1^{\mathrm{hom}}\nabla u_0 - \lambda\big(u_0 + \langle u_1\rangle_\Omega\big) = \langle f\rangle_\Omega, \quad x \in S,$$

$$-\Delta_\omega u_1(x,\omega) - \lambda\big(u_0(x) + u_1(x,\omega)\big) = f(x,\omega), \quad \omega \in \mathcal{O}.$$

$$\xi \cdot a_1^{\mathrm{hom}}\xi = \inf_{\psi \in H^1(\Omega\setminus\mathcal{O})} \int_{\Omega\setminus\mathcal{O}} (\xi + \nabla_\omega\psi)\cdot a_1(\xi + \nabla_\omega\psi)\,dP$$

$$-\nabla\cdot a_1^{\mathrm{hom}}\nabla u_0(x)-\lambda\big(u_0(x)+\langle u_1\rangle_\Omega\big)=\langle f\rangle_\Omega,$$

$$\Delta_\omega u_1(x,\omega)-\lambda\big(u_0(x)+u_1(x,\omega)\big)=f(x,\omega).$$

Decoupling (S_0^ε consists of identical inclusions):

$$u_1(x,\omega)=\lambda u_0(x)v(\omega)$$

$$-\Delta_\omega v=\lambda v+1.$$

$$-\mathrm{div}\, a_1^{\mathrm{hom}}\nabla u_0=\beta(\lambda)u_0$$

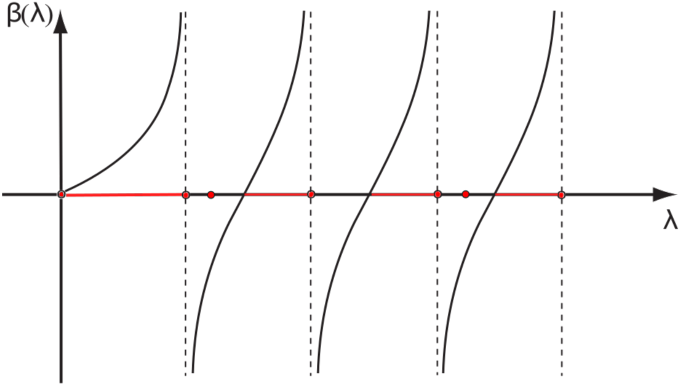
where

$$\beta(\lambda):=\lambda\big(1+\lambda\langle v\rangle_\Omega\big)=\lambda+\lambda^2\sum_{j=1}^{\infty}\frac{\langle\varphi_j\rangle_y^2}{\nu_j-\lambda}P\big(\{\widetilde{\omega}:\widetilde{\omega}_0=1\}\big)$$

is a stochastic version of the "Zhikov's β-function".

$$\sigma(\mathcal{A}_{\mathrm{hom}})=\{\lambda:\beta(\lambda)\geq 0\}\cup\{\nu_1',\nu_2',\ldots\}\qquad(\text{if } S=\mathbb{R}^n)$$

Zhikov's β-function



$$-\nabla\cdot a_1^{\rm hom}\nabla u_0(x)-\lambda\big(u_0(x)+\langle u_1\rangle_\Omega\big)=\langle f\rangle_\Omega,$$

$$\Delta_\omega u_1(x,\omega)-\lambda\big(u_0(x)+u_1(x,\omega)\big)=f(x,\omega).$$

Decoupling (S_0^ε consists of scaled inclusions of the same geometrical shape, $Y_r:=rY$, $r\in[r_1,r_2]$):

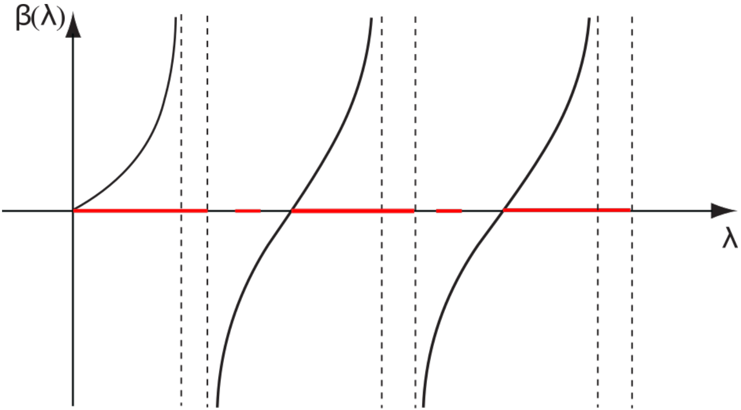
$$u_1(x,\omega)=\lambda u_0(x)v(\omega)$$

$$-\Delta_\omega v=\lambda v+1.$$

$$\beta(\lambda):=\lambda\big(1+\lambda\langle v\rangle_\Omega\big)=\lambda+\lambda^2\int\limits_{\{\widetilde\omega_0\in[r_1,r_2]\}}\sum_{j=1}^\infty\frac{\langle\varphi_{j,\widetilde\omega_0}\rangle_y^2}{\nu_{j,\widetilde\omega_0}-\lambda}\,dP(\widetilde\omega).$$

$$\sigma(\mathcal{A}_{\rm hom})=\{\lambda:\beta(\lambda)\ge0\}\cup\{\nu_{1,r},\nu_{2,r},\dots\}\cup\{\nu'_{1,r},\nu'_{2,r},\dots\}\qquad(\text{if } S=\mathbb{R}^n)$$

Zhikov's β-function



Thank you!