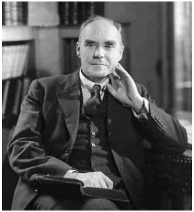
AVERAGING AND INITIAL LAYER ANALYSIS IN PASSIVE TRANSPORT

Harsha Hutridurga

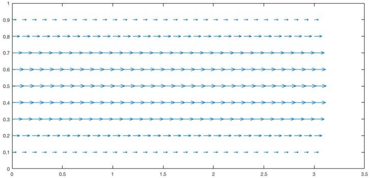
Imperial College London

19 June 2019 - Recent Advances in Homogenisation theory



- G.I. TAYLOR posed an **interesting question**.
- Take a tube filled with fluid.
- Study spreading of dissolved solutes inside the tube.
- Interplay between molecular diffusion and advection

$$\partial_t c + \mathbf{b} \cdot \nabla c - d\Delta c = 0.$$



- Scenarios $d \equiv 0$ or $\mathbf{b} \equiv$ constant **do not mix well**.
- Shearing advective field: **Poiseuille** flow in a tube.
- Empirical formula for **Effective diffusion**.

[Ref.] G.I.TAYLOR, **Proc. Roy. Soc. Lond. A Math.**, *Vol 219 (1953).*
[Ref.] G.K.BATCHELOR, **The life and legacy of G.I.Taylor**, *(1996).*

MOTIVATION

- Quantity of interest: certain CONCENTRATION FIELD.
- Evolving under the influence of
  - advection by an incompressible field.
  - molecular diffusion.
- Physical quantity immersed in a fluid flow
  - temperature (heat).
  - concentration of some solute.
- Chlorophyll moved around in ocean.
- Heat evolving on the surface of the ocean.

[Ref.] EARTH OBSERVATORY WEBPAGE OF NASA FOR CERTAIN GLOBAL MAPS

https://earthobservatory.nasa.gov

[Media.] TEMPERATURE AND CHLOROPHYLL MAPS (2002-2016)

INTRICATE CONNECTIONS

♣ Other scientific disciplines such as
  - Geophysics: oceanography
  - Engineering: chemical engineering
  - Biology: motor proteins

♣ Particular applications:
  - weak heat fluctuations in fluids
  - dyes used to visualising flow patterns
  - pollutants dispersing in the environment
  - gas exchange in the lungs
  - blood circulation

[Ref.] A.MAJDA, P.KRAMER, *Phy. Rep. (1999).*

INTERESTING SERIES OF LECTURES ON YOUTUBE

[Ref.] JEAN-LUC THIFEAULT, **Stirring and Mixing**, *Geophysical Phenomena, ICTS (2016).*

ADVECTION-DIFFUSION EQUATION

Initial-boundary value problem for the **scalar** unknown $u^\varepsilon(t,x)$

$$\partial_t u^\varepsilon + \frac{1}{\varepsilon}\mathbf{b}\cdot\nabla u^\varepsilon - \Delta u^\varepsilon = 0 \qquad \text{in } \mathbb{R}_+ \times \Omega,$$

$$u^\varepsilon(0,x) = u^{\text{in}}(x) \qquad \text{in } \Omega,$$

$$\nabla u^\varepsilon \cdot \mathbf{n}(x) = 0 \qquad \text{on } \mathbb{R}_+ \times \partial\Omega.$$

- ♣ $\varepsilon > 0$ is a parameter (to regulate strength of advective field).
- ♣ Molecular diffusion **weak** compared to the strength of advection.

PASSIVE SCALAR

- ♣ Advective field $\mathbf{b}(x)$ is a datum.
- ♣ **Neglect buoyancy**: no feedback.

ADVECTIVE FIELD

Advective field $\mathbf{b}(x) : \Omega \to \mathbb{R}^d$ is such that

- ♣ It is **prescribed** – We are **not** solving fundamental fluid equation.
- ♣ It is **incompressible** (divergence-free, solenoidal), i.e.,

$$\nabla \cdot \mathbf{b}(x) = 0 \qquad \text{for a.e. } x \in \Omega.$$

- ♣ It has **zero normal flux** at the boundary, i.e.,

$$\mathbf{b}(x) \cdot \mathbf{n}(x) = 0 \quad \text{for a.e. } x \in \partial\Omega.$$

- ♣ It is as **smooth** as the computations demand.

ADDITIONAL COMMENT

- ♣ In case of a **feedback**: coupled system for $u^\varepsilon(t, x)$ and $\mathbf{b}(x)$.
- ♣ RAYLEIGH-BÉNARD convection system – more complicated.

- Consider two-dimensional incompressible Navier-Stokes equations

$$\partial_t \mathbf{u} + (\mathbf{u} \cdot \nabla)\, \mathbf{u} + \nabla p = \varepsilon \Delta \mathbf{u}$$
$$\nabla \cdot \mathbf{u} = 0$$

- Introduce the vorticity $\omega := \nabla \times \mathbf{u}$ – satisfies the coupled system

$$\partial_t \omega + \mathbf{u} \cdot \nabla \omega = \varepsilon \Delta \omega$$
$$\mathbf{u} = \nabla^{\perp} \Psi = \left(-\partial_{x_2} \Psi, \partial_{x_1} \Psi\right)$$
$$-\Delta \Psi = \omega$$

- Long-time scaling yields

$$\partial_t \omega + \frac{1}{\varepsilon} \mathbf{u} \cdot \nabla \omega = \Delta \omega$$

- This is a long term goal...

Interesting Questions

$$\partial_t u^\varepsilon + \frac{1}{\varepsilon}\mathbf{b}\cdot\nabla u^\varepsilon - \Delta u^\varepsilon = 0$$

♣ Relaxation to equilibrium

- In the evolution for $u^\varepsilon(t,x)$ **how long** does it take to equilibrate?
- If we wait long enough, will we reach an **uniform** temperature?
- Is the **rate** of convergence **uniform** in ε?
- Are there **special** advective fields which result in **quicker equilibration**?

♣ Strong advection limit

- Does there exist a **limit point** for the sequence $\{u^\varepsilon(t,x)\}$?
- If $\lim_{\varepsilon\to 0} u^\varepsilon = \overline{u}$, how can we **characterise** the limit $\overline{u}$?
- Is there a **rate** of convergence in terms of ε?
- What is the **interplay** with the **rate** of convergence and the chosen **advective field**?

THIS TALK IS DERIVED FROM

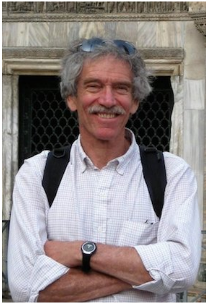
[Ref.] T.HOLDING, H.H, J.RAUCH, **SIAM J. Math. Anal.**, *Vol 49, No 1, pp. 222–271 (2017).*

[Ref.] T.HOLDING, H.H, J.RAUCH, **in preparation**, *(2018).*

Access to both the arXiv and published versions on my webpage:

`http://hutridurga.wordpress.com`

SCIENTIFIC COLLABORATORS



Joint work with T. Holding (Imperial), J. Rauch (Michigan)

- Consider the **null space** and the **range space** of $\mathbf{b} \cdot \nabla$

$$\mathcal{N}_{\mathbf{b}} := \left\{ v \in \mathrm{L}^2(\Omega) \text{ s.t. } \operatorname{div}(\mathbf{b} v) = 0 \text{ in the sense of distributions} \right\}.$$

$$\mathcal{W}_{\mathbf{b}} := \left\{ \mathbf{b} \cdot \nabla v \text{ for } v \in \mathrm{H}^1(\Omega) \right\} \subset \mathrm{L}^2(\Omega).$$

- Hilbert's theorem yields **orthogonal decomposition**

$$\mathrm{L}^2(\Omega) = \mathcal{N}_{\mathbf{b}} \oplus \overline{\mathcal{W}_{\mathbf{b}}}$$

i.e., for any $v \in \mathrm{L}^2(\Omega)$, there exists a **unique decomposition**

$$v = v_n + v_r$$

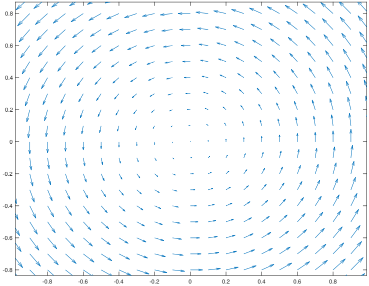
such that $v_n \in \mathcal{N}_{\mathbf{b}}$ and $v_r \in \mathcal{N}_{\mathbf{b}}^{\perp} = \overline{\mathcal{W}_{\mathbf{b}}}$

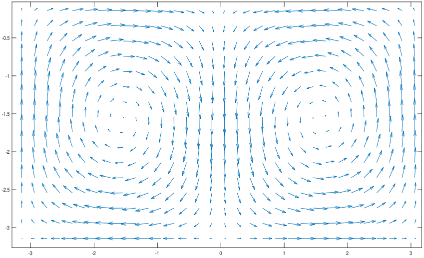
- **Projection** on to $\mathcal{N}_{\mathbf{b}}$ denoted $\mathcal{P} : \mathrm{L}^2(\Omega) \mapsto \mathcal{N}_{\mathbf{b}}$

♣ A rotation in two dimensions

$$\mathbf{b}(x_1, x_2) = \begin{pmatrix} -x_2 \\ x_1 \end{pmatrix}$$

♣ $x_1^2 + x_2^2 \in \mathcal{N}_{\mathbf{b}}$





♣ A two dimensional cellular flow

$$\mathbf{b}(x_1, x_2) = \begin{pmatrix} -\sin(x_1)\cos(x_2) \\ \cos(x_1)\sin(x_2) \end{pmatrix}$$

♣ $\sin(x_1)\sin(x_2) \in \mathcal{N}_{\mathbf{b}}$

- For any $v \in \mathrm{L}^2(\Omega)$, the projection $\mathcal{P}v$ can be **computed** as

$$\|v - \mathcal{P}v\|_{\mathrm{L}^2(\Omega)} = \min_{g \in \mathcal{N}_{\mathbf{b}}} \|v - g\|_{\mathrm{L}^2(\Omega)}$$

- More useful way to interpret the projection map $\mathcal{P}$ is due to

von Neumann's ergodic theorem:

For any $v \in \mathrm{L}^2(\Omega)$,

$$\mathcal{P}v(x) := \lim_{\ell \to \infty} \frac{1}{2\ell} \int_{-\ell}^{\ell} v\left(\Phi_\tau(x)\right) \mathrm{d}\tau$$

with the **flow** $\Phi_\tau(x) : \mathbb{R} \times \Omega \to \Omega$ defined as

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}\tau} \Phi_\tau(x) & = \mathbf{b}\left(\Phi_\tau(x)\right) \\ \Phi_0(x) & = x \end{cases}$$

Theorem (HOLDING, H, RAUCH (2018))

Let $u^\varepsilon(t,x)$ be the solution to the initial-boundary value problem. Then

$$u^\varepsilon \rightharpoonup \overline{u} \qquad \textit{weakly in } \mathrm{L}^2$$

with $\overline{u}(t,x)$ being the unique solution to

$$\partial_t \overline{u} - \Delta \overline{u} = g \in \mathcal{N}_{\mathbf{b}}^{\perp}$$
$$\overline{u}(t,\cdot) \in \mathcal{N}_{\mathbf{b}}$$
$$\overline{u}(0,\cdot) = \mathcal{P}u^{\mathrm{in}}(\cdot)$$
$$\nabla \overline{u}(t,x)\cdot \mathbf{n}(x) = 0 \qquad \textit{on } \mathbb{R}_+ \times \partial\Omega.$$

- The condition $\overline{u}(t,\cdot) \in \mathcal{N}_{\mathbf{b}}$ is treated as a **constraint**.
- The source $g(t,\cdot) \in \mathcal{N}_{\mathbf{b}}^{\perp}$ is the associated **Lagrange multiplier**.
- The **initial datum** has got **projected** on to the null space $\mathcal{N}_{\mathbf{b}}$.

$$\partial_t u^\varepsilon + \frac{1}{\varepsilon}\mathbf{b}\cdot\nabla u^\varepsilon - \Delta u^\varepsilon = 0$$

- By L^2-weak convergence, we mean that for any $\psi(t,x) \in \mathrm{L}^2$,

$$\lim_{\varepsilon\to 0} \iint u^\varepsilon(t,x)\psi(t,x)\,\mathrm{d}x\,\mathrm{d}t = \iint \overline{u}(t,x)\psi(t,x)\,\mathrm{d}x\,\mathrm{d}t$$

- For a weak convergence, we cannot give a rate of convergence
- Limit evolution with the constraint should be interpreted as

solving **heat equation** on the subspace $\mathcal{N}_{\mathbf{b}}$.

- Can we improve it to strong convergence (then we explore the rate)

$$\lim_{\varepsilon\to 0} \|u^\varepsilon - \overline{u}\|_{\mathrm{L}^2} = 0.$$

- Are there advective fields $\mathbf{b}(x)$ which result in strong convergence?
- One possible obstacle for such a result is

$$u^\varepsilon(0,x) = u^{\mathrm{in}}(x); \qquad \overline{u}(0,x) = \mathcal{P}u^{\mathrm{in}}(x).$$

STRATEGY

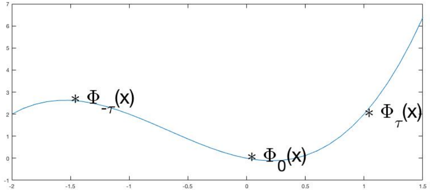
♣ WORK IN A MOVING FRAME OF REFERENCE

- Rather than studying $u^\varepsilon(t,x)$ in a **fixed frame**,
- we study u^ε taken along a **moving frame**.
- **Dynamics** of the moving frame dictated by the field $\mathbf{b}(x)$.

MEASURE PRESERVING FLOW

♣ Consider the **flow**
$\Phi_\tau(x) : \mathbb{R} \times \Omega \to \Omega$
defined as

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}\tau}\Phi_\tau(x) & = \mathbf{b}\left(\Phi_\tau(x)\right) \\ \Phi_0(x) & = x \end{cases}$$



- Rather than studying $u^\varepsilon(t,x)$ we study the family

$$u^\varepsilon\left(t, \Phi_{t/\varepsilon}(x)\right)$$

- Flow is evaluated at $\frac{t}{\varepsilon}$
- We introduce a fast time variable $\tau := \frac{t}{\varepsilon}$

$$t = \mathcal{O}(\varepsilon) \implies \tau = \mathcal{O}(1).$$

- For any $\tau \in \mathbb{R}$, the map $x \mapsto \Phi_\tau(x)$ defines a **change-of-variable**
- Associated **Jacobian matrix**

$$J(\tau,x) = \begin{bmatrix} \frac{\partial \Phi^1_\tau}{\partial x_1} & \cdots & \frac{\partial \Phi^1_\tau}{\partial x_d} \\ \vdots & & \vdots \\ \frac{\partial \Phi^d_\tau}{\partial x_1} & \cdots & \frac{\partial \Phi^d_\tau}{\partial x_d} \end{bmatrix} = \left(\frac{\partial \Phi^i_\tau}{\partial x_j}\right)^d_{i,j=1}$$

- $\mathbf{b}(x)$ incompressible $\implies$ flow $\Phi_\tau(x)$ is **volume preserving**,

$$\text{i.e., } \det\left(J(\tau,x)\right) = 1 \qquad \text{for all } \tau \in \mathbb{R}.$$

Theorem (HOLDING, H, RAUCH (2017))

Let $u^\varepsilon(t,x)$ *be the solution to the initial-boundary value problem. Suppose Jacobian matrix* $J(\cdot,x) \in \mathcal{A}$*, an* **algebra with mean value**. *Then for each* $t > 0$*,*

$$\lim_{\varepsilon \to 0} \left\| u^\varepsilon(t,\cdot) - u_0\left(t, \Phi_{-t/\varepsilon}(\cdot)\right) \right\|_{\mathrm{L}^2(\Omega)} = 0$$

where $u_0(t,\mathrm{X})$ *solves a* **diffusion equation**

$$\partial_t u_0 = \nabla_{\mathrm{X}} \cdot \left(\mathfrak{D}(\mathrm{X}) \nabla_{\mathrm{X}} u_0 \right); \qquad u_0(0,\mathrm{X}) \;\; = u^{\mathrm{in}}(\mathrm{X})$$

with the **diffusion matrix** *given by*

$$\mathfrak{D}(\mathrm{X}) = \lim_{\ell \to \infty} \frac{1}{2\ell} \int_{-\ell}^{+\ell} J(\tau,\mathrm{X})^\top J(\tau,\mathrm{X})\, \mathrm{d}\tau.$$

- Computing the time derivative

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big[u^\varepsilon\left(t,\Phi_{t/\varepsilon}(x)\right)\Big] = \partial_t u^\varepsilon\left(t,\Phi_{t/\varepsilon}(x)\right) + \frac{1}{\varepsilon}\frac{\mathrm{d}}{\mathrm{d}t}\Phi_{t/\varepsilon}(x)\cdot\nabla_X u^\varepsilon\left(t,\Phi_{t/\varepsilon}(x)\right)$$
$$= \partial_t u^\varepsilon\left(t,\Phi_{t/\varepsilon}(x)\right) + \frac{1}{\varepsilon}\mathbf{b}\left(\Phi_{t/\varepsilon}(x)\right)\cdot\nabla_X u^\varepsilon\left(t,\Phi_{t/\varepsilon}(x)\right)$$

- RHS is the advection term taken along the flow $\Phi_{t/\varepsilon}(x)$.
- X denotes the Lagrangian coordinate.
- Computing the spatial derivative

$$\nabla\Big[u^\varepsilon\left(t,\Phi_{t/\varepsilon}(x)\right)\Big] = {}^{\top}J\left(\frac{t}{\varepsilon},x\right)\nabla_X u^\varepsilon\left(t,\Phi_{t/\varepsilon}(x)\right)$$

  where ${}^{\top}\cdot$ denotes transpose.
- Note the **dependance** of Jacobian on the **fast time variable**.

- Need to compute the Laplacian term along the flow $\Phi_{t/\varepsilon}(x)$.
- Consider the associated energy

$$\int_\Omega \langle \nabla u^\varepsilon(t,x), \nabla u^\varepsilon(t,x)\rangle \, \mathrm{d}x$$

- Perform the change of variables $x \mapsto \Phi_{t/\varepsilon}(x)$ inside the integral

$$\int_\Omega \left\langle {}^\top J\left(\frac{t}{\varepsilon}, x\right) \nabla_X u^\varepsilon\left(t, \Phi_{t/\varepsilon}(x)\right), {}^\top J\left(\frac{t}{\varepsilon}, x\right) \nabla_X u^\varepsilon\left(t, \Phi_{t/\varepsilon}(x)\right) \right\rangle \underbrace{\frac{\mathrm{d}X}{|\det(J)|}}_{=1}$$

- Hence the Laplacian along the flow $\Phi_{t/\varepsilon}(x)$ becomes

$$\nabla_X \cdot \left(J\left(\frac{t}{\varepsilon}, x\right) {}^\top J\left(\frac{t}{\varepsilon}, x\right) \nabla_X u^\varepsilon\left(t, \Phi_{t/\varepsilon}(x)\right)\right)$$

- We have seen that $u^\varepsilon(t,x)$ solves

$$\partial_t u^\varepsilon + \frac{1}{\varepsilon}\mathbf{b}\cdot\nabla u^\varepsilon - \Delta u^\varepsilon = 0$$

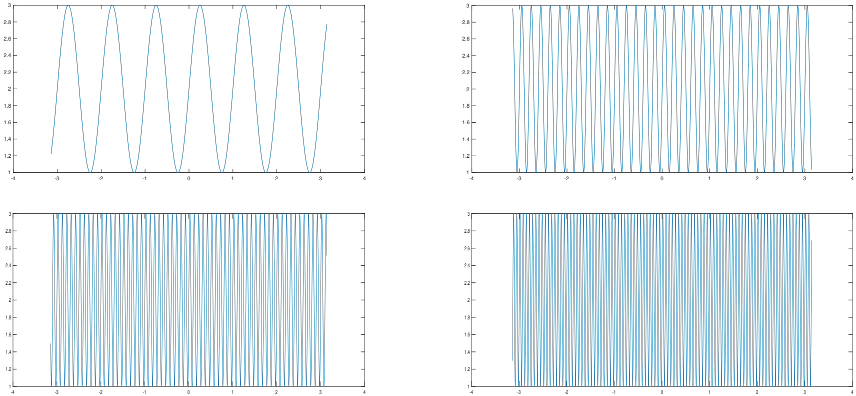
if and only if $u^\varepsilon(t, \Phi_{t/\varepsilon}(x))$ solves

$$\partial_t u^\varepsilon - \nabla_X \cdot \left(J\left(\frac{t}{\varepsilon}, x\right)^{\top} J\left(\frac{t}{\varepsilon}, x\right) \nabla_X u^\varepsilon \right) = 0$$

- Pass to the limit as $\varepsilon \to 0$ in the weak formulation.
- $\nabla_X u^\varepsilon$ weakly converges in L^2
- If the family $J^{\top}J\left(\frac{t}{\varepsilon}, x\right)$ strongly converges, we are good because

$$f^\varepsilon \rightharpoonup f_0, \quad h^\varepsilon \to h_0 \implies f^\varepsilon h^\varepsilon \to f_0\, h_0 \quad \text{in } \mathcal{D}'$$

♣ Take $f_n(t) = 2 + \sin(2n\pi t)$ over $[-\pi, \pi]$ with n being a parameter.



♣ f_n cannot converge in almost any point.

Lemma

Suppose $f \in \mathrm{L}^\infty(\mathbb{R})$. *Define the dilated sequence*

$$f^\varepsilon(t) := f\left(\frac{t}{\varepsilon}\right).$$

$$\textit{If} \qquad f^\varepsilon \rightharpoonup M(f) \qquad \textit{weakly} \ * \ \textit{in}\ \mathrm{L}^\infty(\mathbb{R}) \qquad \textit{as}\ \varepsilon \to 0$$

where $M(f)$ *is a finite constant. Then, the limit is characterised as*

$$M(f) = \lim_{\ell\to\infty} \frac{1}{2\ell} \int_{-\ell}^{\ell} f(\tau)\,\mathrm{d}\tau.$$

♣ By $h^\varepsilon \rightharpoonup h_0$ weakly $*$ in $\mathrm{L}^\infty(\mathbb{R})$, we mean

$$\lim_{\varepsilon\to 0} \int_{\mathbb{R}} h^\varepsilon(t)\psi(t)\,\mathrm{d}t = \int_{\mathbb{R}} h_0(t)\psi(t)\,\mathrm{d}t \quad \forall \psi \in \mathrm{L}^1.$$

PRODUCT OF TWO WEAKLY CONVERGING SEQUENCES

$$\sin\left(\frac{2\pi t}{\varepsilon}\right) \rightharpoonup 0; \qquad \text{But} \quad \int_0^1 \sin^2\left(\frac{2\pi t}{\varepsilon}\right)\, \mathrm{d}t \to \frac{1}{2}$$

QUINTESSENTIAL TOUGH QUESTION IN ANALYSIS

- Passing to the limit in product of weakly converging sequences
- This is the question of interest in

Homogenization theory of differential equations.

- A typical problem in homogenization is to study

$$v^\varepsilon(x) \in \mathrm{H}^1_0(\Omega)$$

$$-\nabla \cdot \left(\mathfrak{a}\left(\frac{x}{\varepsilon}\right)\nabla v^\varepsilon\right) = g$$

in the $\varepsilon \ll 1$ regime.

- Usually we make some **structural** assumption on the coefficient $\mathfrak{a}$
- Homogenization motivates some structural assumption on $J(\cdot, x)$

Notation: $\mathcal{B}(\mathbb{R})$ - space of bounded continuous functions.

Definition (Algebra with mean value)

$\mathcal{A}$ be a Banach subalgebra of $\mathcal{B}(\mathbb{R})$ with following properties:

- ♣ *$\mathcal{A}$ contains the* **constants**.
- ♣ *$\mathcal{A}$ is* **translation invariant**, *i.e. $f(\cdot - a) \in \mathcal{A}$ whenever $f \in \mathcal{A}$.*
- ♣ *Any $f \in \mathcal{A}$ possesses a* **mean value** *in the following sense*

$$f\left(\frac{\cdot}{\varepsilon}\right) \rightharpoonup M(f) \quad \text{in } L^\infty(\mathbb{R})\text{-weak* as } \varepsilon \to 0.$$

We have already seen that

$$M(f) = \lim_{\ell \to \infty} \frac{1}{2\ell} \int_{-\ell}^{\ell} f(\tau)\, \mathrm{d}\tau.$$

[Ref.] V.V.JIKOV, E.V.KRIVENKO, **Matem. Zametki** *(1983)*.

[Ref.] V.V.JIKOV, S.M.KOZLOV, O.A.OLEINIK, **Springer-Verlag** *(1994)*.

Example (Periodic functions)

$\mathcal{A} = \mathcal{C}_{\text{per}}$ *be space of continuous functions* **periodic** *with period* 1.

$$M(u) = \int_0^1 u(\tau)\,\mathrm{d}\tau.$$

Example (Functions that converge at infinity)

$\mathcal{A}$ *be space of continuous functions that converge to a limit at infinity*

$$M(u) = \lim_{|\tau|\to\infty} u(\tau).$$

Example (Almost-periodic functions)

♣ $\mathsf{T}(\mathbb{R})$ *be the set of all trigonometric polynomials, i.e. all* $u(t)$ *that are finite linear combinations of functions in the set*

$$\Big\{ \cos(kt), \sin(kt) : k \in \mathbb{R} \Big\}.$$

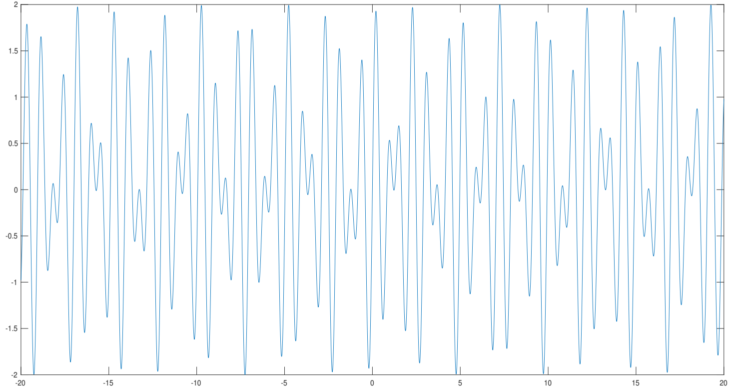
The space of almost-periodic functions in the sense of **Bohr** *is the closure of* $\mathsf{T}(\mathbb{R})$ *in the supremum norm,*

i.e., given a $\delta > 0$ *and an almost-periodic function* $u(t)$*,*
there exists a $g(t) \in \mathsf{T}(R)$ *s.t.*

$$\|u(\cdot) - g(\cdot)\|_{\mathrm{L}^\infty} < \delta.$$

TYPICAL EXAMPLE OF AN ALMOST-PERIODIC FUNCTION

$$\sin(2\pi t) + \sin(2\sqrt{2}\pi t)$$



- ♣ Fix an arbitrary algebra w.m.v. $\mathcal{A}$.
- ♣ Take $\mathbf{b}(x)$ such that Jacobian matrix $J(\cdot, x) \in \mathcal{A}$, i.e., in particular

$$\sup_{\tau \in \mathbb{R}} |J(\tau, x)| < \infty$$

A NEW NOTION OF WEAK CONVERGENCE

Definition (Σ-convergence along flow(HHR-2017))

A family $\{u^\varepsilon\} \subset \mathrm{L}^2((0,\ell)\times\Omega)$ *is said to* **Σ-converge along the flow** Φ_τ *to a limit* $u_0(t, X, s) \in \mathrm{L}^2((0,\ell)\times\Omega\times\Delta(\mathcal{A}))$ *if, for any smooth test function* $\psi(t, x, \cdot) \in \mathcal{A}$, *we have*

$$\lim_{\varepsilon\to 0} \iint_{(0,\ell)\times\Omega} u^\varepsilon(t,x)\psi\left(t, \Phi_{-t/\varepsilon}(x), \frac{t}{\varepsilon}\right) \mathrm{d}x\, \mathrm{d}t$$

$$= \iiint_{(0,\ell)\times\Omega\times\Delta(\mathcal{A})} u_0(t, X, s)\widehat{\psi}(t, X, s)\, \mathrm{d}\beta(s)\, \mathrm{d}X\, \mathrm{d}t.$$

Example (Constant drift)

$$\mathbf{b}(x) = \overline{\mathbf{b}} \in \mathbb{R}^d.$$

Jacobian $J(\cdot)$ **identity** *for all times.*

Example (Asymptotically constant drift)

$$\mathbf{b}(x) = \begin{cases} \mathbf{b}^* & \text{when} \quad x_1 < -a, \\ \mathbf{c}(x) & \text{when} \quad x_1 \in [-a, a], \\ \mathbf{b}^{**} & \text{when} \quad x_1 > a, \end{cases}$$

- ♣ $a > 0$, $\mathbf{e}_1 \cdot \mathbf{b}^*, \mathbf{e}_1 \cdot \mathbf{b}^{**} > 0$
- ♣ $\mathbf{c}(x)$ *chosen to make* $\mathbf{b}$ *continuously differentiable.*
- ♣ *Any integral curve spends only* **finite time** T *in* $\{x_1 \in [-a, a]\}$.

Example (Euclidean motions)

$$\mathbf{b}(x) = \mathbf{A}x + \bar{\mathbf{b}} \qquad \text{with } \mathbf{A} = -{}^{\top}\mathbf{A} \quad \text{and} \quad \bar{\mathbf{b}} \in \mathbb{R}^d.$$

♣ *Associated flow*

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\Phi_\tau(x) = \mathbf{A}\Phi_\tau(x) + \bar{\mathbf{b}}; \qquad \Phi_0(x) = x.$$

♣ *Jacobian $J(\cdot, x)$ is an orthogonal matrix.*

♣ *Jacobian matrix has no growth in τ.*

Example

Let $\Omega \subset \mathbb{R}^2$ *and* $\Omega := B(0;1)$. *Advective field is a rigid rotation*

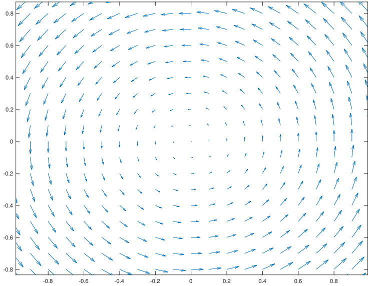
$$\mathbf{b}(x_1, x_2) = \begin{pmatrix} -x_2 \\ x_1 \end{pmatrix}$$

- Associated flow

$$\Phi^1_\tau(x_1, x_2) = -x_2 \sin\tau + x_1 \cos\tau$$
$$\Phi^2_\tau(x_1, x_2) = x_1 \sin\tau + x_2 \cos\tau$$

- Jacobian matrix

$$J(\tau, x_1, x_2) = \begin{bmatrix} \cos\tau & -\sin\tau \\ \sin\tau & \cos\tau \end{bmatrix}$$



- algebra w.m.v. $\mathcal{A} = \mathcal{C}_{\text{per}}$.
- Note that $J^\top J = \text{Id}$.
- Hence diffusion $\mathfrak{D} = \text{Id}$.

♣ For any incompressible field $\mathbf{b}(x)$, family $u^\varepsilon(t,x)$ converges weakly

$$\lim_{\varepsilon\to 0} \iint_{(0,\ell)\times\Omega} u^\varepsilon(t,x)\psi(t,x)\,\mathrm{d}x\,\mathrm{d}t = \iint_{(0,\ell)\times\Omega} \overline{u}(t,x)\psi(t,x)\,\mathrm{d}x\,\mathrm{d}t \quad \forall\psi\in \mathrm{L}^2$$

with $\overline{u}$ solves an evolution equation with **constraint** $\overline{u}(t,\cdot)\in\mathcal{N}_{\mathbf{b}}$

♣ For any field $\mathbf{b}(x)$ such that $J(\cdot,x)\in\mathcal{A}$, a certain **algebra w.m.v.**

Then, for any $t\in(0,\ell)$ we have

$$\lim_{\varepsilon\to 0} \|w^\varepsilon(t,x) - u_0(t,x)\|_{\mathrm{L}^2(\Omega)} = 0$$

where $w^\varepsilon(t,x) := u^\varepsilon\left(t, \Phi_{t/\varepsilon}(x)\right)$ and

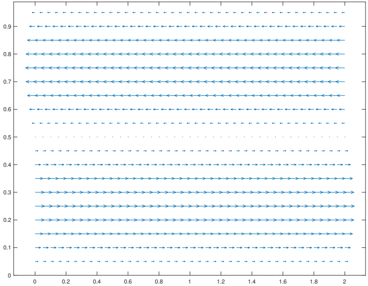
u_0 solves a diffusion equation with diffusivity $\mathfrak{D}$.

♣ **There is no contradiction.**

The operative phrase being **moving frame**.

- Two dimensional shear flow

$$\mathbf{b}(x) = \begin{pmatrix} a(x_2) \\ 0 \end{pmatrix}$$



- Measure preserving flow

$$\Phi_\tau(x_1, x_2) = \begin{pmatrix} x_1 + a(x_2)\tau \\ x_2 \end{pmatrix}$$

- Jacobian matrix

$$J(\tau, x_1, x_2) = \begin{bmatrix} 1 & a'(x_2)\tau \\ 0 & 1 \end{bmatrix}$$

- Compute $J(\tau, x)^\top J(\tau, x)$

$$= \begin{bmatrix} 1 & a'(x_2)\tau \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ a'(x_2)\tau & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 + |a'(x_2)|^2\tau^2 & a'(x_2)\tau \\ a'(x_2)\tau & 1 \end{bmatrix}$$

- Not uniformly bounded in τ
main difficulty: $M(J) \not< \infty$.
- Lagrangian stretching.

- Find a **weight function** $\omega(\tau)$ such that

$$\omega^2(\tau)J(\tau,x)^\top J(\tau,x) \qquad \text{has a mean value.}$$

- Take $\omega(\tau) = \left(1+\tau^2\right)^{-\frac{1}{2}}$, then

$$\omega^2(\tau)J(\tau,x)^\top J(\tau,x) = \begin{bmatrix} \frac{1+|a'(x_2)|^2\tau^2}{1+\tau^2} & \frac{a'(x_2)\tau}{1+\tau^2} \\ \frac{a'(x_2)\tau}{1+\tau^2} & \frac{1}{1+\tau^2} \end{bmatrix}$$

- Mean value does exist

$$\lim_{\ell\to\infty} \frac{1}{2\ell}\int_{-\ell}^{\ell} \omega^2(\tau)J(\tau,x)^\top J(\tau,x)\,\mathrm{d}\tau = \begin{bmatrix} |a'(x_2)|^2 & 0 \\ 0 & 0 \end{bmatrix}$$

- Note that the resulting limit matrix is **not of full rank**.

- We saw earlier that $u^\varepsilon(t, \Phi_{t/\varepsilon}(x))$ solves

$$\partial_t u^\varepsilon - \nabla_X \cdot \left(J\left(\frac{t}{\varepsilon}, x\right)^\top J\left(\frac{t}{\varepsilon}, x\right) \nabla_X u^\varepsilon \right) = 0$$

Definition (Initial layer time variable)

Introduce a new time variable via the ode

$$\frac{\mathrm{d}\tau(t;\varepsilon)}{\mathrm{d}t} = \frac{1}{\omega^2(t/\varepsilon)} = 1 + \frac{t^2}{\varepsilon^2}; \qquad \tau(0,\varepsilon) = 0.$$

- Rather than looking at $u^\varepsilon(t, \Phi_{t/\varepsilon}(x))$, consider $u^\varepsilon(\tau(t;\varepsilon), \Phi_{t/\varepsilon}(x))$ which solves

$$\frac{1}{\omega^2(t/\varepsilon)} \partial_\tau u^\varepsilon \left(\tau(t;\varepsilon), \Phi_{t/\varepsilon}(x)\right)$$
$$= \nabla_X \cdot \left(J\left(\frac{t}{\varepsilon}, x\right)^\top J\left(\frac{t}{\varepsilon}, x\right) \nabla_X u^\varepsilon \left(\tau(t;\varepsilon), \Phi_{t/\varepsilon}(x)\right) \right)$$

- Explicit integration yields

$$\mathcal{T}(t;\varepsilon) = t + \frac{t^3}{3\varepsilon^2}$$

- Note that for $\varepsilon \ll 1$, we have

$$t \sim \varepsilon^{\frac{2}{3}}\, \mathcal{T}^{\frac{1}{3}}$$

- Pass to the limit as $\varepsilon \to 0$ (at least formally)

$$\partial_{\mathcal{T}} u_0(\mathcal{T}, \mathcal{X}) = |a'(\mathcal{X}_2)|^2 \partial^2_{\mathcal{X}_1} u_0(\mathcal{T}, \mathcal{X}); \qquad u(0, \mathcal{X}_1, \mathcal{X}_2) = u^{\text{in}}(\mathcal{X}_1, \mathcal{X}_2)$$

- Equation is **degenerate** – diffusion occurs only in $\mathcal{X}_1$ direction.
- Diffusion occurs **along the direction of the flow**.
- **Long time behaviour** of the solution $u_0(\mathcal{T}, \mathcal{X}_1, \mathcal{X}_2)$: for each $\mathcal{X}_2$

$$\lim_{\mathcal{T}\to\infty} \int \left| u_0(\mathcal{T}, \mathcal{X}_1, \mathcal{X}_2) - \mathcal{P}u^{\text{in}}(\mathcal{X}_2) \right|^2 \mathrm{d}\mathcal{X}_1 = 0.$$

- Initial layer dynamics has projected the initial datum on to $\mathcal{N}_{\mathbf{b}}$
- Recall the evolution equation with constraint and projected datum

$$\partial_t \overline{u} - \Delta \overline{u} = g \in \mathcal{N}_{\mathbf{b}}^{\perp}$$
$$\overline{u}(t,\cdot) \in \mathcal{N}_{\mathbf{b}}$$
$$\overline{u}(0,\cdot) = \mathcal{P} u^{\mathrm{in}}(\cdot)$$

- In case of the **shear flow**: initial datum $\mathcal{P} u^{\mathrm{in}}(x_2)$
- In case of the **shear flow**: constraint $\overline{u}(t,\cdot) \in \mathcal{N}_{\mathbf{b}} \implies \overline{u} \equiv \overline{u}(t, x_2)$
- The evolution equation then becomes

$$\partial_t \overline{u} - \partial_{x_2}^2 \overline{u} = g \in \mathcal{N}_{\mathbf{b}}^{\perp}$$

- At times of $\mathcal{O}(1)$ – diffusion orthogonal to flow lines

- For a stream function $H(x_1, x_2)$, consider

$$\mathbf{b}(x) = \nabla^{\perp} H = \begin{pmatrix} -\partial_{x_2} H \\ \partial_{x_1} H \end{pmatrix}$$

- ∇H and $\nabla^{\perp} H$ are orthogonal away from fixed points of H.

Lemma (HOLDING, H, RAUCH (2017))

Let x be a periodic point of the flow with period $P(x)$. Then,

$$J(\tau, x)\nabla^{\perp} H(x) = (\nabla^{\perp} H)(\Phi_{-\tau}(x)),$$

$$J(\tau, x)\nabla H(x) = (\nabla^{\perp} H)(\Phi_{-\tau}(x)) \left[\frac{(\nabla P(x) \cdot \nabla H(x))}{P(x)} \tau + f(\tau, x) \right] + \\ + \frac{|\nabla H(x)|^2}{|(\nabla H)(\Phi_{-\tau}(x))|^2} (\nabla H)(\Phi_{-\tau}(x)),$$

where $f(\cdot, x)$ is a continuous $P(x)$-periodic function.

HAMILTONIAN FLOWS

- ♣ Enhanced relaxation along the flow in a time-boundary layer.
- ♣ At times of $\mathcal{O}(1)$ – diffusion orthogonal to flow lines.
- ♣ A close link to results in Freidlin-Wentzell theory.

[Ref.] M.FREIDLIN, A.WENTZELL, **Springer-Verlag** *(1998)*.

STRONG CONVERGENCE

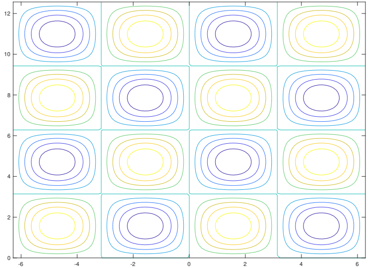
Theorem (HOLDING, H, RAUCH (2018))

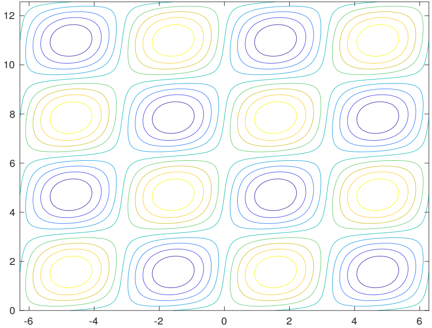
Let $\mathbf{b}(x) = \nabla^\perp H$ *for some* **non-degenerate** $2D$ *Hamiltonian. Let* $u^\varepsilon(t,x)$ *be the solution family to the* **initial-boundary value problem**. *Let* $\overline{u}(t,x)$ *be the solution to the evolution equation with the* **constraint**. *Then we have*

$$\lim_{\varepsilon \to 0} \|u^\varepsilon - \overline{u}\|_{\mathrm{L}^2((0,\ell)\times\Omega)} = 0.$$

- Cellular flows
(Taylor stream function)

$$H(x_1, x_2) = \sin(x_1)\sin(x_2)$$





- Cat's eye flows

$$H(x_1, x_2) = \sin(x_1)\sin(x_2) + \delta\cos(x_1)\cos(x_2)$$

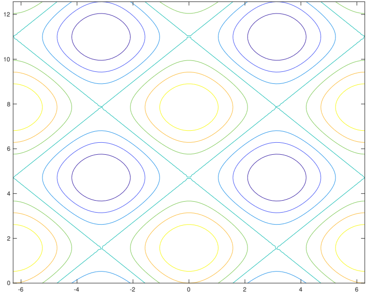
SOME EXAMPLES OF ADVECTIVE FIELDS

- ABC flows
(ARNOLD-BELTRAMI-CHILDRESS)

$$\begin{pmatrix} \mathbf{b}_1 \\ \mathbf{b}_2 \\ \mathbf{b}_3 \end{pmatrix} = \begin{pmatrix} A\sin(z) + C\cos(y) \\ B\sin(x) + A\cos(z) \\ C\sin(y) + B\cos(x) \end{pmatrix}$$

- Take $(A, B, C) = (0, 1, 1)$

$$\begin{pmatrix} \mathbf{b}_1 \\ \mathbf{b}_2 \\ \mathbf{b}_3 \end{pmatrix} = \begin{pmatrix} \cos(y) \\ \sin(x) \\ \sin(y) + \cos(x) \end{pmatrix} = \begin{pmatrix} \partial_y H \\ -\partial_x H \\ H(x,y) \end{pmatrix}$$

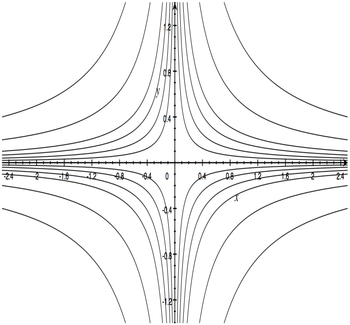


- Hamiltonian

$$H(x,y) = \sin(y) + \cos(x)$$

$$\begin{pmatrix} \mathbf{b}_1 \\ \mathbf{b}_2 \end{pmatrix} = \begin{pmatrix} -\lambda x_1 \\ \lambda x_2 \end{pmatrix} \qquad \text{i.e., with } H(x_1, x_2) = \lambda\, x_1 x_2.$$

$$\left(\Phi^1_\tau, \Phi^2_\tau\right)(x_1, x_2) = \left(e^{-\lambda\tau} x_1,\, e^{\lambda\tau} x_2\right); \qquad J(\tau) = \begin{pmatrix} e^{-\lambda\tau} & 0 \\ 0 & e^{\lambda\tau} \end{pmatrix}$$



- Considering **microscopic oscillations** in fluid fields

$$\mathbf{b}\left(x, \frac{x}{\varepsilon}, \frac{x}{\varepsilon^2}, \cdots, \frac{x}{\varepsilon^n}\right)$$

collaboration with G.PAVLIOTIS (IMPERIAL).

- Stochastic homogenization with **random solenoidal fields**

$$\mathbf{b}\left(\frac{x}{\varepsilon}, \omega\right)$$

collaboration with S.NEUKAMM, M.SCHÄFFNER (DRESDEN).

- Numerical illustration of the time-boundary layer phenomenon

using adaptive wavelet galerkin method

collaboration with R.STEVENSON (AMSTERDAM).

- High contrast in diffusivity
collaboration with K.CHEREDNICHENKO (BATH), S.COOPER (DURHAM).

Proposition (long time behaviour)

There exists a uniform constant $\gamma > 0$ *such that*

$$\left\| u^\varepsilon(t,\cdot) - \langle u^{\rm in}\rangle \right\|_{\mathrm{L}^2(\Omega)} \lesssim e^{-\gamma\, t}$$

where $\langle u^{\rm in}\rangle$ *denotes the average*

$$\langle u^{\rm in}\rangle := \frac{1}{|\Omega|}\int_\Omega u^{\rm in}(x)\,\mathrm{d}x$$

Multiply the evolution by $u^\varepsilon(t,x)$ and integrate over the spatial domain

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega |u^\varepsilon(t,x)|^2\,\mathrm{d}x + \frac{1}{2\varepsilon}\int_\Omega \mathbf{b}(x)\cdot\nabla\,|u^\varepsilon(t,x)|^2\,\mathrm{d}x + \int_\Omega |\nabla u^\varepsilon(t,x)|^2\,\mathrm{d}x = 0$$

$$\text{i.e.,}\qquad \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega |u^\varepsilon(t,x)|^2\,\mathrm{d}x = -\int_\Omega |\nabla u^\varepsilon(t,x)|^2\,\mathrm{d}x.$$

Result follows by **Poincaré** inequality and **Grönwall's** inequality.

Definition (Relaxation enhancing fields)

An incompressible field $\mathbf{b}(x)$ *is called relaxation enhancing if for any* $\delta > 0$*, there exists* $\overline{\varepsilon}(\delta) > 0$ *such that* $\forall \varepsilon$ *with* $\varepsilon < \overline{\varepsilon}(\delta)$ *we have*

$$\left\| u^\varepsilon(1,\cdot) - \langle u^{\text{in}} \rangle \right\|_{L^2(\Omega)} < \delta.$$

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Theorem (Constantin et al. (2008))

An incompressible field $\mathbf{b}(x)$ *is relaxation enhancing* **if and only if**

$$\mathcal{N}_{\mathbf{b}} \cap \mathrm{H}^1(\Omega) \quad \textit{has } \textbf{no non-trivial elements}.$$

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