

High-contrast homogenization of high-contrast PDE systems

Valery Smyshlyaev

University College London, UK

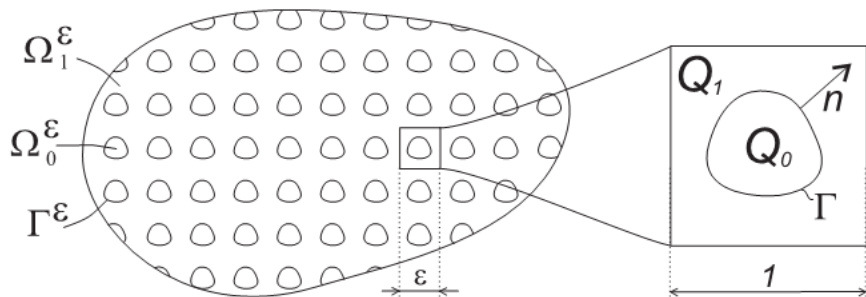
June 18, 2018

(partly joint with Ilia Kamotski UCL, and Shane Cooper Durham)

Outline:

- Classical vs high-contrast (= 'microresonant') homogenization, from a spectral theoretic point of view (re V.V. Zhikov 2000, and followers).
- **Effects**: band gaps, dispersion, 'negative' materials, etc.
- **Analysis**: ('two-scale') resolvent convergence /semigroups/ spectral convergence, (two-scale) compactness, etc.
- **'Partial' degeneracies** and 'generalised' micro-resonances (more of effects/ applications); A general theory for PDE systems under a generic 'decomposition' assumption: I. Kamotski & V.S., *Applicable Analysis* 2018, a special issue in memory of V.V. Zhikov.
- Failure of the above decomposition assumption vs two-scale convergence w.r.t. **measures** (Zhikov 2000): example of highly anisotropic fibers from Cherednichenko, S., Zhikov'06.
- From two-scale convergence to error bounds: Error bounds for EVs and EFs for a defect in a high-contrast medium, Kamotski & S., *J. Math. Sci N.Y.* 2018, another special issue in memory of V.V. Zhikov.
- + Some current developments.

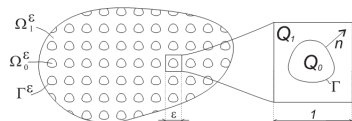
Classical Periodic Homogenisation (= 'moderate' contrast & low frequencies)



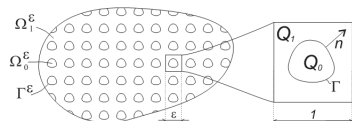
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$$\rho^\varepsilon(x) u_{tt} - \operatorname{div}(a^\varepsilon(x) \nabla u^\varepsilon) = 0 \text{ in } \Omega \subseteq \mathbb{R}^d$$
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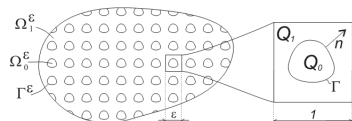
$$a^\varepsilon(x) = a(x/\varepsilon), \quad \rho^\varepsilon = \rho(x/\varepsilon)$$

$$a(y), \rho(y) \text{ } Q\text{-periodic in } y;$$

$$Q = [0, 1]^d \text{ unit cube.}$$

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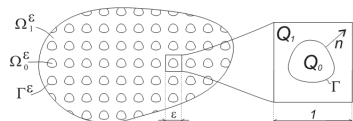
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Asymptotic expansion:

$$u^\varepsilon(x, t) \sim u^0(x, x/\varepsilon, t) + \varepsilon u^1(x, x/\varepsilon, t) + \varepsilon^2 u^2(x, x/\varepsilon, t) + \dots$$

$u^0(x, y, t), u^1(x, y, t), \dots$ Q -periodic in y

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The Homogenisation Theorem

The solution $u^\varepsilon(x, t)$ of the original problem converges, in appropriate norms, to the solution $u_0(x, t)$ of the homogenised problem, as $\varepsilon \rightarrow 0$.

Spectral Theory point of view: resolvent convergence

$A^\varepsilon u = -\operatorname{div}(a^\varepsilon(x)\nabla u)$ is a non-negative self-adjoint operator in Hilbert space $H = L^2(\Omega)$; Ω e.g. bounded Lipschitz (with Dirichlet or Neumann B.C.) or $\Omega = \mathbb{R}^d$.

Let $\lambda > 0$ and let $u^\varepsilon \in H_0^1(\Omega)$ be the unique solution of

$$-\operatorname{div}(a^\varepsilon(x)\nabla u^\varepsilon) + \lambda u^\varepsilon = f^\varepsilon \iff u^\varepsilon = R_\lambda^\varepsilon f^\varepsilon,$$

$R_\lambda^\varepsilon = (A^\varepsilon + \lambda I)^{-1}$ the resolvent. Then

(classical) Theorem (weak/ strong resolvent convergence):

Let $f^\varepsilon, f_0 \in L^2(\Omega)$, $f^\varepsilon \rightarrow f_0$ weakly/ strongly in L^2 . Then $\forall \lambda > 0$,

$$R_\lambda^\varepsilon f^\varepsilon \rightarrow R_\lambda^{\operatorname{hom}} f_0 \text{ (weakly/ strongly in } L^2),$$

where $R_\lambda^{\operatorname{hom}} = (A^{\operatorname{hom}} + \lambda I)^{-1}$ is the resolvent of the homogenised (non-negative, self-adjoint) operator $A^{\operatorname{hom}} = -\operatorname{div}(a^{\operatorname{hom}}\nabla \cdot)$.

N.B. In fact, for $\Omega = \mathbb{R}^d$ or Ω bounded $C^{1,1}$ a (much) stronger operator convergence with an estimate holds, Birman & Suslina ≥ 2000 .

$$\left\| R_\lambda^\varepsilon - R_\lambda^{\operatorname{hom}} \right\|_{L^2 \rightarrow L^2} \leq C\varepsilon.$$

Classical periodic homogenisation: Implications of the resolvent convergence

The strong resolvent convergence implies:

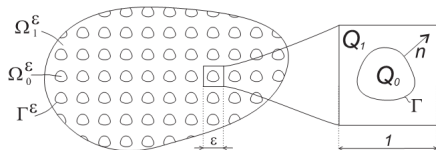
- Strong convergence of spectral projectors;
- Strong convergence of (parabolic) semigroups (Trotter-Kato theorem), and hence of associated parabolic Cauchy problems;
- Convergence of hyperbolic semigroups (and hence of hyperbolic Cauchy problems);
- 'Part of' Hausdorff convergence of spectra:
 $\lambda_0 \in \sigma(A^{hom}) \Rightarrow \exists \lambda^\varepsilon \in \sigma(A^\varepsilon)$ such that $\lambda^\varepsilon \rightarrow \lambda^0$.

Additionally, often (if e.g. Ω bounded Dirichlet), 'spectral compactness' holds (by separate means):

$$\lambda^\varepsilon \in \sigma(A^\varepsilon) \text{ and } \lambda^\varepsilon \rightarrow \lambda_0 \Rightarrow \lambda_0 \in \sigma(A^{hom}).$$

(Implying hence a 'full' convergence of spectra.)

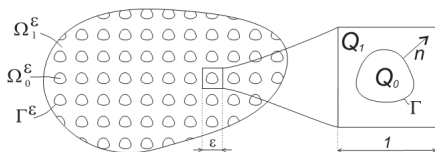
High-contrast homogenization and 'non-classical' two-scale limits (Zhikov 2000, 2004)



$$A^\epsilon u = -\operatorname{div}(a^\epsilon(x)\nabla u^\epsilon)$$

$$a^\epsilon(x) = \begin{cases} \delta & \text{on } \Omega_0^\epsilon \text{ ('soft' phase)} \\ 1 & \text{on } \Omega_1^\epsilon \text{ ('stiff' phase)} \end{cases}$$

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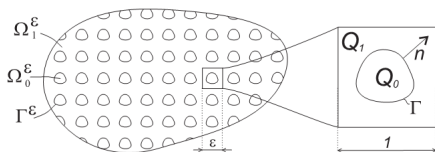


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Contrast $\delta \sim \epsilon^2$ is a **critical scaling** giving rise to 'non-classical' effects (Khruslov 1980s; Arbogast, Douglas, Hornung 1990; Panasenko 1991; Allaire 1992; Sandrakov 1999; Brienne 2002; Bourget, Mikelic, Piatnitski 2003; Bouchitte & Felbaq 2004, ...): elliptic, spectral, parabolic, hyperbolic, nonlinear, non-periodic/ random,

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WHY?

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- **Resonant inclusions:** $\lambda_i \sim \varepsilon$.
- $\lambda_m \sim 1$ (**Macroscale**) $\Rightarrow \lambda_m/\lambda_i \sim \varepsilon^{-1} \Rightarrow \delta \sim \varepsilon^2$

$$a^\varepsilon(x) = \begin{cases} \varepsilon^2 & \text{on } \Omega_0^\varepsilon \text{ (inclusions)} \\ 1 & \text{on } \Omega_1^\varepsilon \text{ (matrix)} \end{cases}$$

Two-scale formal asymptotic expansion:

$\operatorname{div}(a^\varepsilon(x)\nabla u^\varepsilon) + \lambda \rho u^\varepsilon = 0$ (time harmonic waves)

$\iff A^\varepsilon u^\varepsilon = \lambda u^\varepsilon, \lambda = \rho \omega^2$ (spectral problem).

Seek $u^\varepsilon(x) \sim u^0(x, x/\varepsilon) + \varepsilon u^1(x, x/\varepsilon) + \dots u^j(x, y)$ Q-periodic in y .

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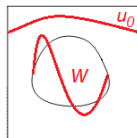
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THEN:

Two-scale limit problem (Zhikov 2000, 2004)

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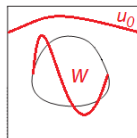
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(u_0, w) , $w(x, y) := u_0(x) + v(x, y)$, solves the **two-scale limit spectral problem** ($\longleftrightarrow A^0 u^0(x, y) = \lambda u^0$):

$$-\operatorname{div}_x(A^{hom} \nabla_x u_0(x)) = \lambda u_0(x) + \lambda \langle v \rangle_y(x) \quad \text{in } \Omega$$

$$\begin{aligned} -\Delta_y v(x, y) &= \lambda(u_0(x) + v(x, y)) && \text{in } Q_0 \\ v(x, y) &= 0 && \text{on } \partial Q_0, \end{aligned}$$

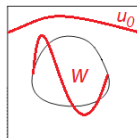
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Uncouple it ↓

Two-scale limit spectral problem

Decouple by choosing $v(x, y) = \lambda u_0(x) b(y)$

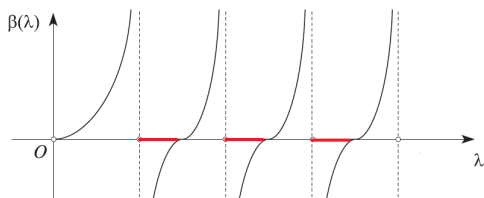
$$\begin{aligned} -\Delta_y b(y) - \lambda b &= 1 && \text{in } Q_0 \\ b(y) &= 0 && \text{on } \partial Q_0 \end{aligned}$$

$$-\operatorname{div}_x(A^{\text{hom}} \nabla u_0(x)) = \beta(\lambda) u_0(x), \quad \text{in } \Omega,$$

$$\text{where } \beta(\lambda) = \lambda + \lambda^2 \langle b \rangle = \lambda + \lambda^2 \sum_{j=1}^{\infty} \frac{\langle \phi_j \rangle_y^2}{\lambda_j - \lambda},$$

(λ_j, ϕ_j) Dirichlet eigen-values/functions of $-\Delta_y$ in inclusion Q_0 (= "micro-resonances"): $\beta < 0$: band gaps (Zhikov 2000);

$\beta(\lambda) = \mu(\omega) < 0 \iff$ "negative density/ magnetism" (Bouchitté & Felbacq, 2004), etc.



Analysis: Two-scale Convergence

Definition

1. Let $u_\varepsilon(x)$ be a bounded sequence in $L^2(\Omega)$. We say (u_ε) weakly two-scale converges to $u_0(x, y) \in L^2(\Omega \times Q)$, denoted by $u_\varepsilon \xrightarrow{2} u_0$, if for all $\phi \in C_0^\infty(\Omega)$, $\psi \in C_\#^\infty(Q)$

$$\int_{\Omega} u_\varepsilon(x) \phi(x) \psi\left(\frac{x}{\varepsilon}\right) dx \longrightarrow \int_{\Omega} \int_Q u_0(x, y) \phi(x) \psi(y) dx dy$$

as $\varepsilon \rightarrow 0$.

2. We say (u_ε) strongly two-scale converges to $u_0 \in L^2(\Omega \times Q)$, denoted by $u_\varepsilon \xrightarrow{2} u_0$, if for all $v_\varepsilon \xrightarrow{2} v_0(x, y)$,

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as $\varepsilon \rightarrow 0$. (implies convergence of norms upon sufficient regularity)

Analysis (Zhikov 2000, 2004):

1. Two-scale limit operator:

A_0 self-adjoint in $H \subset L^2(\Omega \times Q)$, with a band-gap spectrum $\sigma(A_0)$.

$H = L^2(\Omega; \mathbb{C} + L^2(Q_0))$. The (closed, non-negative, densely defined) form for A_0 on

$U = \{u(x, y) = u_0(x) + v(x, y) : u_0 \in H_0^1(\Omega), v \in L^2(\Omega; H_0^1(Q_0))\} \subset H$:

$$\beta(u, u) = \int_{\Omega} A^{hom} \nabla_x u_0 \cdot \overline{\nabla_x u_0} dx + \int_{\Omega} \int_{Q_0} |\nabla_y v(x, y)|^2 dy dx,$$

with domain $D(A_0) \subset U$.

Then, e.g. for $\Omega = \mathbb{R}^d$, the spectrum of A^0 is:

$$\sigma(A^0) = \{\lambda \geq 0 : \beta(\lambda) \geq 0\} \cup_{j=1}^{\infty} \lambda_j^D(Q_0)$$

Analysis (Zhikov 2000, 2004):

2. Two-scale ('pseudo'-)resolvent convergence:

$\forall \lambda > 0, \quad A^\varepsilon u^\varepsilon + \lambda u^\varepsilon = f^\varepsilon \in L^2(\Omega); \quad u^\varepsilon \in H_0^1(\Omega).$

If $f^\varepsilon \xrightarrow{2} f_0(x, y)$ then $u^\varepsilon \xrightarrow{2} u_0(x, y)$. (If $f^\varepsilon \xrightarrow{2} f_0(x, y)$ then $u^\varepsilon \xrightarrow{2} u_0(x, y)$.)

Here u_0 solves "two-scale limit resolvent problem" $A_0 u_0 + \lambda u_0 = P_H f_0$.

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3. Spectral band gaps: (Let $\Omega = \mathbb{R}^d$) cf also Hempel & Lienau 2000

$\sigma(A^\varepsilon) \rightarrow \sigma(A_0)$ in the sense of Hausdorff. (Hence a **Band-gap effect**:

For small enough ε , A^ε has (the smaller ε the more) gaps.

The proof follows from the above two-scale resolvent convergence

+ (additionally) "two-scale spectral compactness".

Analysis (Zhikov 2000, 2004):

2. Two-scale ('pseudo'-)resolvent convergence:

$$\forall \lambda > 0, \quad A^\varepsilon u^\varepsilon + \lambda u^\varepsilon = f^\varepsilon \in L^2(\Omega); \quad u^\varepsilon \in H_0^1(\Omega).$$

If $f^\varepsilon \xrightarrow{2} f_0(x, y)$ then $u^\varepsilon \xrightarrow{2} u_0(x, y)$. (If $f^\varepsilon \xrightarrow{2} f_0(x, y)$ then $u^\varepsilon \xrightarrow{2} u_0(x, y)$.)

Here u_0 solves "two-scale limit resolvent problem" $A_0 u_0 + \lambda u_0 = P_H f_0$.

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N.B. Cherednichenko & Cooper (Arch Rat Mech Anal 2015) have improved the above *strong* two-scale resolvent convergence to an *operator* convergence, with an appropriate 'corrector' B^ε :

$$(A^\varepsilon + \alpha I)^{-1} \rightarrow (A_0 + B^\varepsilon + \alpha I)^{-1}$$

'Frequency' gaps and time-nonlocality (memory):

$$-\operatorname{div}_x(A^{hom}\nabla u_0(x)) = \beta(\omega)u_0(x)$$

(macroscopic) Dispersion relation: $u_0 = e^{ik\cdot x - i\omega t} \Rightarrow$

$$A^{hom}k \cdot k = \beta(\omega)$$

Since A^{hom} positive definite, iff $\beta(\omega) > 0$ waves propagate in **any direction** (iff $\beta(\omega) < 0$ no propagation in any direction \iff Band gap).

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Nonlinearity (= dispersion)/ sign-changing of $\beta(\omega) \rightarrow$
Fourier Transform $\omega \rightarrow \mathbf{t} \rightarrow$ **time-nonlocality** (= 'memory')

$$\int_{-\infty}^t K(t-t')u_{tt}(x, t')dt' - \operatorname{div}_x(A^{hom}\nabla u(x, t)) = 0.$$

'Frequency' vs "directional" gaps and 'partial' degeneracies:

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A question:

Can one similarly get a **spatial nonlocality**? E.g. something like

$$- \operatorname{div}_x \left(\int_{\mathbb{R}^n} \mathcal{A}^{hom}(x - x') \nabla u(x', t) dx' \right).$$

If so, via (inverse) Fourier Transform $\mathbf{x} \longrightarrow \mathbf{k}$, we can, in particular, similarly expect a "spatial" dispersion/ 'negativity'/ 'gaps'.

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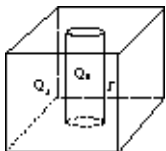
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Yes, we can: e.g. $\mathbf{t} \longrightarrow \mathbf{x}_{n+1}$ then $\omega \longrightarrow \mathbf{k}_{n+1}$ etc, \downarrow

'Frequency' vs "directional" gaps and 'partial' degeneracies

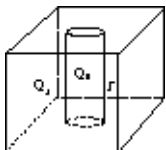
Cherednichenko, V.S., Zhikov (2006): **spatial nonlocality** for homogenised limit with **highly anisotropic** fibers.



$$a^\varepsilon(x) = \begin{cases} \sim 1 & \text{in } Q_1 \text{ (matrix)} \\ \sim \varepsilon^2 & \text{in } Q_0 \text{ "across" fibers} \\ \sim 1 & \text{in } Q_0 \text{ "along" fibers} \end{cases}$$

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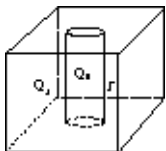
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Then (after uncoupling the two-scale limit system) there is an additional spatially nonlocal macroscopic term "along the fibers" (x_3 -direction), of the form

$$-\frac{\partial}{\partial x_3} \left(\int_{\mathbb{R}} \mathcal{A}^{hom}(x_3 - x'_3) \frac{\partial}{\partial x_3} u(x_1, x_2, x'_3, t) dx'_3 \right).$$

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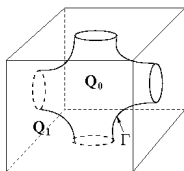
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Notice, in the above fibres, $a^\varepsilon(x) = a^{(1)}(x/\varepsilon) + \varepsilon^2 a^{(0)}(x/\varepsilon)$, where

$$a^{(1)} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \text{i.e. is partially degenerate.}$$

Macroscopic “directional” vs frequency gaps

More generally (V.S. 2009; Linear Elasticity case:)

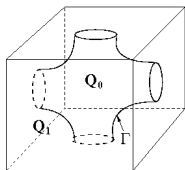


$$C^\varepsilon(x) = \begin{cases} C^1(x/\varepsilon), & x \in Q_1^\varepsilon \\ \varepsilon^2 C^0(x/\varepsilon) + C^2(x/\varepsilon), & x \in Q_0^\varepsilon \end{cases},$$

with (as quadratic forms on symmetric matrices) $C^1, C^0 > \nu I$; but $C^2 \geq 0$ (i.e. possibly ‘partially degenerate’).

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with (as quadratic forms on symmetric matrices) $C^1, C^0 > \nu I$; but $C^2 \geq 0$ (i.e. possibly ‘partially degenerate’).

Then “**directional gaps**” can occur (via formal asymptotic expansions): for certain frequency ranges macroscopic waves can propagate in some directions (e.g. along the fibers above) but cannot in others (e.g. orthogonal to the fibers). ↓

Macroscopic “directional” vs frequency gaps

Macroscopic Dispersion relation: $u = e^{i\mathbf{k}\cdot\mathbf{x} - i\omega t} A$, $|\mathbf{n}| = 1, k > 0 \Rightarrow$

$$\det \left[k^2 \left(C^{hom}(\mathbf{n}) + \gamma(\mathbf{n}, k, \omega) \right) - \omega^2 \beta(\mathbf{n}, k, \omega) \right] = 0, \quad (*)$$

where $C_{ip}^{hom}(k) = C_{ijpq}^{hom} k_j k_q$ (acoustic tensor for ‘half-perforated’ C^{hom});

$$\gamma(\mathbf{n}, k, \omega) = \langle C^2(\mathbf{n})\zeta \rangle, \quad \beta(\mathbf{n}, k, \omega) = \langle \rho \rangle + \langle \rho_0 \zeta \rangle,$$

and $\zeta(y, \mathbf{n}, k, \omega) = \zeta_{ir} = (\zeta^r)_i$ is an elastic (partially degenerating) analog of v , with ζ^r solving in the ‘soft space’

$$V = \left\{ v \in (H_{0,\#}^1(Q_0))^3 \mid C^2 \nabla v = 0 \right\},$$

$$\int_{Q_0} C^0 \nabla \zeta^r \cdot \nabla \eta + k^2 C^2(\mathbf{n}) \zeta^r \cdot \eta - \omega^2 \rho_0 \zeta^r \cdot \eta \, dy = \int_{Q_0} \omega^2 \rho_0 \eta_r - k^2 C^2(\mathbf{n}) \eta \, dy, \quad \forall \eta \in V.$$

Examples (V.S. 2009): (*) giving a ‘directional localization’.

Other examples of 'partial degeneracies':

- 'Easy to shear hard to compress' elastic inclusions:

$$\mu_j \sim \varepsilon^2, \quad \lambda_j \sim 1 \text{ (Shane Cooper, 2013.)}$$

- Photonic crystal fibers for a 'near critical' propagation constant (S. Cooper, I. Kamotski, V.S.: arxiv 2014).
- 3-D Maxwell with high electric permittivity (non-magnetic) inclusions (Cherednichenko, Cooper, 2015): $\epsilon_j \sim \varepsilon^{-2}$

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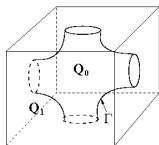
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Some interesting effects in all the above, due to the 'partial degeneracies'.

Analysis: General 'Partial' Degeneracies (I. Kamotski and V.S., a special issue in memory of V.V. Zhikov, *Applicable Analysis*, 2018)



Consider a resolvent problem:

$$\Omega \subset \mathbb{R}^d, \quad \lambda > 0,$$

$$-\operatorname{div}(a^\varepsilon(x) \nabla u^\varepsilon) + \lambda \rho^\varepsilon u^\varepsilon = f^\varepsilon \in L^2(\Omega),$$

$$u^\varepsilon \in (H_0^1(\Omega))^n, \quad n \geq 1.$$

A general degeneracy:

$$a^\varepsilon(x) = a^{(1)}\left(\frac{x}{\varepsilon}\right) + \varepsilon^2 a^{(0)}\left(\frac{x}{\varepsilon}\right), \quad a^{(l)} \in \left(L_\#^\infty(Q)\right)^{n \times d \times n \times d}, \quad a_{ijpq} = a_{pqij};$$

$$a_{ijpq}^{(1)}(y) \zeta_{ij} \zeta_{pq} \geq 0, \quad \forall \zeta \in \mathbb{R}^{n \times d}; \quad a^{(1)} + a^{(0)} > \nu I : \text{ ("strong ellipticity")}$$

$$\int_{\mathbb{R}^d} (a^{(1)} + a^{(0)})(y) \nabla w \cdot \nabla w \geq \nu \|\nabla w\|_{L^2(\mathbb{R}^d)}^2, \quad \forall w \in H^1(\mathbb{R}^d).$$

$$\rho^\varepsilon(x) = \rho\left(\frac{x}{\varepsilon}\right), \quad \rho \in \left(L_\#^\infty(Q)\right)^{n \times n}, \quad \rho_{ij} = \rho_{ji}, \quad \rho > \nu I.$$

Two-scale formal asymptotic expansion:

$$u^\varepsilon(x) \sim u^0(x, x/\varepsilon) + \varepsilon u^1(x, x/\varepsilon) + \dots \quad u^j(x, y) \text{ } Q\text{-periodic in } y.$$

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Then $a^{(1)}(y) \nabla_y u^0(x, y) = 0.$

$$\longleftrightarrow \quad u^0(x, \cdot) \in V := \{ u(u) : a^{(1)}(y) \nabla_y u^0(x, y) = 0 \}$$

Weak formulation:

$$\begin{aligned} \int_{\Omega} \left[a^{(1)} \left(\frac{x}{\varepsilon} \right) \nabla u \cdot \nabla \phi(x) + \varepsilon^2 a^{(0)} \left(\frac{x}{\varepsilon} \right) \nabla u \cdot \nabla \phi(x) + \lambda \rho^\varepsilon(x) u \cdot \phi(x) \right] dx \\ = \int_{\Omega} f^\varepsilon(x) \cdot \phi(x) dx, \quad \forall \phi \in (H_0^1(\Omega))^d. \end{aligned}$$

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A priori estimates:

$$\|u^\varepsilon\|_2 \leq C \|f^\varepsilon\|_2, \quad \|\varepsilon \nabla u^\varepsilon\|_2 \leq C \|f^\varepsilon\|_2, \quad \left\| (a^{(1)}(x/\varepsilon))^{1/2} \nabla u^\varepsilon \right\|_2 \leq C \|f^\varepsilon\|_2.$$

Let $\|f^\varepsilon\|_2 \leq C.$

Weak two-scale limits. Key assumption on the degeneracy

Introduce

$$V := \left\{ v \in (H_{\#}^1(Q))^n \mid a^{(1)}(y) \nabla_y v = 0 \right\}$$

(subspace of “microscopic oscillations”), and

$$W := \left\{ \psi \in (L_{\#}^2(Q))^{n \times d} \mid \operatorname{div}_y \left(\left(a^{(1)}(y) \right)^{1/2} \psi(y) \right) = 0 \text{ in } (H_{\#}^{-1}(Q))^n \right\}$$

(“microscopic fluxes”)

Then, up to a subsequence, $u^\varepsilon \xrightarrow{2} u_0(x, y) \in L^2(\Omega; V)$

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Key assumption:

There exists a constant $C > 0$ such that for all $v \in (H_{\#}^1(Q))^n$
there is $v_1 \in V$ with $\|v - v_1\|_{(H_{\#}^1(Q))^n} \leq C \|a^{(1)}(y) \nabla_y v\|_{L^2}$ (*)

The key assumption examples (more in KS'2018):

The key assumption (*) holds for most of the previously considered cases:

1. Classical homogenization:

$a^{(1)}(y) \geq \nu > 0 \Rightarrow V = \{v = \text{const}\} \Rightarrow (*) \iff$ Poincare inequality
with the mean: $\|v - \langle v \rangle\|_{(H^1_{\#}(Q))^n} \leq C \|\nabla_y v\|_{L^2(Q)}$

2. Double porosity models: $a^{(1)}(y) = \chi_1(y)$ (characteristic function on
'connected' phase Q_1). $\Rightarrow V = \{v = \text{const} + H^1_0(Q_0)\} \Rightarrow (*) \iff$

Extension lemma: $\exists v_1 \in H^1_0$ s.t. $\|v - v_1\|_{(H^1_{\#}(Q))^n} \leq C \|v\|_{H^1(Q_1)}$

3. Elasticity; 'half-soft' inclusions (Cooper 2013).

$V = \{v = \text{const}^3 + (H^1_0(Q_0))^3 : \text{div } v = 0\} \Rightarrow (*) \iff$ 'Modification'
lemma (with a prescribed divergence): $\exists v_1 \in H^1_0(Q_0)$ s.t. $\text{div } v_1 = 0$ and
 $\|\nabla(v - v_1)\|_{(L^2(Q))^n} \leq C \left(\|\nabla v\|_{L^2(Q_1)} + \|\text{div } v\|_{L^2(Q_0)} \right)$

The key assumption examples (continued)

4. Elasticity with stiff fibers/ grains (cf. M. Bellieud, SIAM J Math Anal 2010): $d = n = 3$,

Single stiff cylindrical fiber: $Q_1 = \hat{Q}_1 \times [0, 1)$, $\overline{\hat{Q}_1} \subset [0, 1)^2$.

$$V = \left\{ v \in (H_{\#}^1(Q))^3 : v(y) = c + \alpha y \times e_3 \text{ in } Q_1; \quad c \in \mathbb{R}^3, \alpha \in \mathbb{R} \right\}$$

(\longleftrightarrow translations and **rotations** about the cylinder's axis).

$v \in (H_{\#}^1(Q))^3 \mapsto \tilde{v}(y) = \tilde{c} + \tilde{\alpha} y \times e_3$, where $\tilde{c} \in \mathbb{R}^3$, $\tilde{\alpha} \in \mathbb{R}$ are such that

$$\int_{Q_1} \tilde{v} \, dy = \int_{Q_1} \tilde{v} \cdot (y \times e_3) \, dy = 0.$$

\mapsto can choose $v_1 = v - E\tilde{v}$, where $E : (H_{\#}^1(Q_1))^3 \rightarrow (H_{\#}^1(Q))^3$ is a bounded extension.

Then (*) follows from a Korn-type inequality for 'periodic' cylinders.

Similarly extended to several stiff fibers parallel to different axes and/ or isolated stiff grains, cf. M. Bellieud'10.

The key assumption examples (continued)

5. Photonic crystal fibers (Cooper, I. Kamotski, S.'14):

$$V = \left\{ v \in \left(H_{\#}^1(Q) \right)^2 : v_{1,1} + v_{2,2} = v_{1,2} - v_{2,1} = 0 \text{ in } Q_1 \right\} \text{ (cf$$

Cauchy-Riemann). $\Rightarrow (*) \iff \exists v_1 \in V$ s.t.

$$\|\nabla(v - v_1)\|_{(L^2(Q))^n} \leq C \left(\|v_{1,1} + v_{2,2}\|_{L^2(Q_1)} + \|v_{1,2} - v_{2,1}\|_{L^2(Q_1)} \right)$$

6. 3-D Maxwell with high contrast (cf. Cherednichenko & Cooper, 2015):

$$V = \left\{ v \in \left(H_{\#}^1(Q) \right)^3 : \operatorname{div} v = 0; \operatorname{curl} v = 0 \text{ in (simply connected) } Q_1 \right\}. \Rightarrow$$

$(*) \iff \exists v_1 \in V$ s.t.

$$\|\nabla(v - v_1)\|_{(L^2(Q))^n} \leq C \left(\|\operatorname{curl} v\|_{L^2(Q_1)} + \|\operatorname{div} v\|_{L^2(Q)} \right).$$

7. If $a^{(1)}(y) \equiv a^{(1)}$ (a constant, not depending on y), then $(*)$

$\iff \mathcal{A}$ -quasiconvexity 'constant rank' key decomposition assumption (Fonseca-Mueller).

The two-scale Limit Operator (generally 'non-local')

Let Ω be e.g. bounded Lipschitz, or $\Omega = \mathbb{R}^d$. Introduce $U \subset L^2(\Omega; V)$:

$$U := \left\{ u(x, y) \in L^2(\Omega; V) \mid \exists \xi(x, y) \in L^2(\Omega; W) \text{ s.t.}, \forall \Psi(x, y) \in C^\infty(\Omega; W), \right.$$

$$\left. \int_{\Omega} \int_Q \xi(x, y) \cdot \Psi(x, y) dx dy = - \int_{\Omega} \int_Q u(x, y) \cdot \nabla_x \cdot \left(\left(a^{(1)}(y) \right)^{1/2} \Psi(x, y) \right) \right.$$

Define $T : U \rightarrow L^2(\Omega; W)$ by $Tu := \xi$. **Then, $\xi_0 = Tu_0$** , and

Theorem (Strong two-scale ('pseudo'-)resolvent convergence):

Let $f^\varepsilon \xrightarrow{2} f_0(x, y)$. Then $u^\varepsilon \xrightarrow{2} u_0(x, y)$, uniquely solving:

Find $u_0 \in U$ such that $\forall \phi \in U$

$$\int_{\Omega} \int_Q \left\{ Tu_0(x, y) \cdot T\phi_0(x, y) + a^{(0)}(y) \nabla_y u_0(x, y) \cdot \nabla_y \phi_0(x, y) + \right. \\ \left. + \lambda \rho(y) u_0(x, y) \cdot \phi_0(x, y) \right\} dy dx = \int_{\Omega} \int_Q f_0(x, y) \cdot \phi_0(x, y) dy dx.$$

Two-scale limit self-adjoint operator

The above defines a self-adjoint two-scale limit operator A^0 in Hilbert space $H = \text{closure of } U \text{ in } L^2_\rho(\Omega \times Q)$, with domain $D(A^0) \subset U$:

$$D(A^0) = \{u(x, y) \in U : \exists w \in H \text{ s.t. } \beta(u, v) = (w, v)_H \forall v \in U\};$$

$$\beta(u, v) := \int_{\Omega} \int_Q Tu(x, y) \cdot \overline{Tv(x, y)} + a^{(0)}(y) \nabla_y u(x, y) \cdot \overline{\nabla_y v(x, y)} dy dx$$

Crudely, $A^0 u = T^* T - \text{div}_y (a^{(0)}(y) \nabla_y u)$,

$$T^* T = -P_V \text{div}_x \left(\left(a^{(1)}(y) \right)^{1/2} P_W \left(a^{(1)}(y) \right)^{1/2} \nabla_x u(x, y) \right),$$

$P_W = L^2$ -orthogonal projector on W (admissible micro-fluxes) \longleftrightarrow solving the 'generalized' corrector problem:

$$\text{div}_y \left(a^{(1)}(y) [\nabla_x u(x, y) + \nabla_y u_1(x, y)] \right) = 0,$$

$P_V = L^2$ -orthogonal projector on V (admissible micro-fields).

Implications of the operator convergence

1. Strong two-scale convergence of spectral projectors. (Implies a 'part' of spectral convergence.)
2. Strong two-scale convergence of semigroups (a two-scale analogue of the Trotter-Kato theorem, cf. Zhikov 2000, Zh-Pastukhova 2007):

$$f^\varepsilon \xrightarrow{2} f_0(x, y) \in H \quad \Rightarrow \quad e^{-A^\varepsilon t} f^\varepsilon \xrightarrow{2} e^{-A^0 t} f_0(x, y)$$

Hence, implications for hom-n of double porosity-type (parabolic) prblms:

$$\rho^\varepsilon(x) \frac{\partial u^\varepsilon}{\partial t} - \operatorname{div}(a^\varepsilon(x) \nabla u^\varepsilon) = 0, \quad u^\varepsilon(x, 0) = f^\varepsilon(x),$$

If $f^\varepsilon \xrightarrow{2} f_0(x, y) \in H$, then the (unique) solution $u^\varepsilon \xrightarrow{2} u_0(x, y, t)$, $\forall t \geq 0$, where u_0 is the unique solution of two-scale Cauchy problem:

$$\frac{\partial u_0}{\partial t} + A^0 u_0 = 0, \quad u_0(x, y, 0) = f_0(x, y), \quad (*)$$

Implications (cf e.g. Khruslov & Co 1990s; Zhikov 2000): The limit system (*) holds under most general assumptions, and may generally give macroscopic (multi-phase) 'flows' coupled by not only temporal nonlocality (= memory) but also a 'spatial' one.

Implications of the operator convergence (continued)

2'. Strong two-scale convergence of *hyperbolic* semigroups (cf. Pastukhova 2005):

Implications for homogenisation of degenerating hyperbolic problems:

$$\rho^\varepsilon(x) \frac{\partial^2 u^\varepsilon}{\partial t^2} - \operatorname{div}(a^\varepsilon(x) \nabla u^\varepsilon) = 0, \quad u^\varepsilon(x, 0) = f^\varepsilon(x), \quad u_t^\varepsilon(x, 0) = g^\varepsilon(x),$$

$f^\varepsilon \in H_0^1$, $g^\varepsilon \in L^2$. If (for example) $f^\varepsilon \xrightarrow{2} f_0(x, y) \in U$, $g^\varepsilon \xrightarrow{2} g_0(x, y) \in H$, and

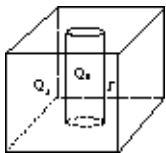
$$\limsup_{\varepsilon \rightarrow 0} \int_{\Omega} a^\varepsilon(x) \nabla f^\varepsilon \cdot \nabla f^\varepsilon < \infty,$$

then, for $T > 0$, the (unique) solution $u^\varepsilon \xrightarrow{2} u^0(x, y, t)$ in $L^2(0, T; L^2(\Omega))$, where u^0 is the unique solution of two-scale Cauchy problem:

$$\frac{\partial^2 u^0}{\partial t^2} + A^0 u^0 = 0, \quad u^0(x, y, 0) = f_0(x, y), \quad u_t^0(x, y, 0) = P g_0(x, y).$$

Examples with the key assumption (*) **not** held

Cherednichenko, V.S., Zhikov (2006): highly anisotropic fibers.



$$a^\varepsilon(x) = \begin{cases} \sim 1 & \text{in } Q_1 \text{ (matrix)} \\ \sim \varepsilon^2 & \text{in } Q_0 \text{ "across" fibers} \\ \sim 1 & \text{in } Q_0 \text{ "along" fibers} \end{cases}$$

Here $d = 3$, $n = 1$, $Q_0 = \hat{Q}_0 \times [0, 1)$, $\overline{\hat{Q}_0} \subset [0, 1)^2$;

$$a^{(1)}(y) = \chi_1(y)I + \chi_0(y) \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \alpha \end{pmatrix}, \quad \alpha > 0.$$

Then

$$V = \left\{ v(y) \in H_{\#}^1(Q) : v(y) = c + \tilde{v}(\tilde{y}), c \in \mathbb{R}, \tilde{v} \in H_0^1(\hat{Q}_0), \tilde{y} = (y_1, y_2) \right\}$$

One can then see that (*) is not held, for e.g.

$$v_n(y) = v_0(\tilde{y}) \sin(ny_1) \cos(2\pi y_3), \quad \tilde{v} \in H_0^1(\hat{Q}_0), \text{ when } n \rightarrow \infty.$$

However the two-scale (pseudo-) resolvent convergence is still held (CSZ'06), via two-scale convergence with respect to **measures**,

$$d\mu_\varepsilon = \chi_1(x/\varepsilon)dx, \text{ cf. Zhikov'00.}$$

On the spectral convergence

The strong (two-scale) resolvent convergence implies:

$$\lambda_0 \in \sigma(A^0) \Rightarrow \exists \lambda^\varepsilon \in \sigma(A^\varepsilon) \text{ such that } \lambda^\varepsilon \rightarrow \lambda_0.$$

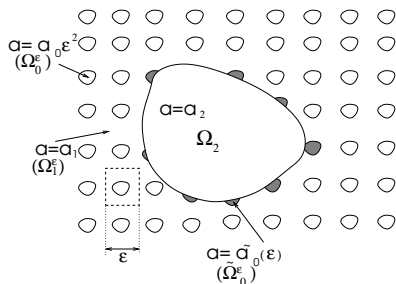
The converse property (spectral compactness) is often desired: if $\lambda^\varepsilon \in \sigma(A^\varepsilon)$ such that $\lambda^\varepsilon \rightarrow \lambda_0$ then $\lambda_0 \in \sigma(A^0)$.

It does not hold in general. However it holds in some particular cases (which then has to be established by separate means), e.g.

- Isolated 'soft' inclusions (Zhikov 2000, 2001).
- Isolated soft elastic inclusions, including 'semisoft' (soft in shear, stiff in compression; Cooper 2013)

Examples when it does not hold, often correspond to an 'inter-connected' soft phase, keeping supporting as $\varepsilon \rightarrow 0$ (microscopically) quasi-periodic Bloch waves, not described by the adopted two-scale (i.e. *periodic* in $y = x/\varepsilon$) framework. Nevertheless, in some cases the approach can be extended to include y -quasi-periodic limits (e.g. in photonic crystal fibers with a pre-critical propagation, Cooper, Kamotski, V.S. 2014; 1-D scalar case Cherednichenko, Cooper, Giennau, 2014).

Error bound for localized defect in a high-contrast medium: I. Kamotski & V.S., J. Math. Sci. N.Y., 2018, Zhikov memorial issue



A spectral problem:

$$A_\epsilon u^\epsilon := -\operatorname{div} a^\epsilon(x) \nabla u^\epsilon = \lambda_\epsilon u^\epsilon, \quad u^\epsilon = u^\epsilon(x)$$

with a two-scale homogenized spectral problem $A^0 u^0 = \lambda^0 u^0$, $u^0 = u^0(x, y)$. **Then**

Error bound for localized defect in a high-contrast medium: I. Kamotski & V.S., J. Math. Sci. N.Y., 2018, Zhikov memorial issue

Then (Theorem, Kamotski & S., 2018):

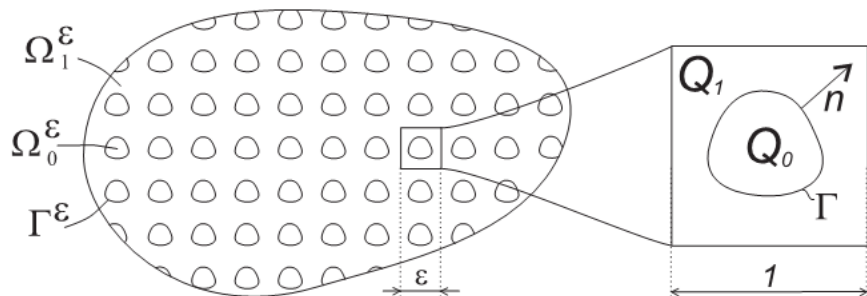
$$|\lambda_\varepsilon - \lambda_0| \leq C \varepsilon^{1/2}$$

$$\|u^\varepsilon - u^0\|_{L^2} \leq C \varepsilon^{1/2}$$

The latter, in combination with the results of M. Cherdantsev (2009) on two-scale resolvent convergence for this problem and a two-scale spectral compactness (provided no ‘resonant’ boundary cut inclusions).

Further announcement: operator-type error bounds for general classes of high-contrast problems (Cooper, I. Kamotski, V.S., 2018)

Dynamic problems with random micro-resonances



Let $\Omega = \mathbb{R}^3$, and $Q_0 = B_{r_0}$ (periodic balls). Consider initial value problem

$$u_{tt}^\varepsilon - \operatorname{div} \left(a^\varepsilon \left(\frac{x}{\varepsilon} \right) \nabla u^\varepsilon \right) = 0, \quad u^\varepsilon(x, 0) = g(x), \quad u_t^\varepsilon(x, 0) = h(x),$$

g, h compactly supported/ rapidly decaying.

Dynamic problems with random micro-resonances

Let $a^\varepsilon(y) = \chi_1(y) + \varepsilon^2 \chi_0(y) a(\omega, m)$, $m \in \mathbb{Z}^3$ I.I.D. with a 'nice' probability density p , $\int_0^\infty p(a) da = 1$.

Then, formal asymptotics gives:

$$u^\varepsilon(x, t) \sim u^0(x, t) + v(x, t, x/\varepsilon; \omega),$$

with (u^0, v) coupled via:

$$u_{tt}^0 + \langle v_{tt} \rangle_{y, \omega} - \operatorname{div} \left(A^{hom} \nabla_x u^0 \right) = 0,$$

$$u_{tt}^0 + v_{tt} - a(\omega, m) \Delta_y v = 0,$$

where A^{hom} is a (classical) homogenized matrix for periodic perforated domain.

Formal uncoupling then gives an equation for u^0 of a form:

Dynamic problems with random micro-resonances

$$|Q_1|u_{tt}^0(x, t) + K_1u_t^0(x, t) - K_2u^0(x, t) - \int_0^\infty \mathcal{K}(\tau)u^0(x, t - \tau) - \operatorname{div} \left(A^{hom} \nabla_x u^0 \right) = 0 \quad (+I.C.s),$$

with rather explicit K_1 , K_2 , and $\mathcal{K}(\tau)$, with “right” signs.

Taking the Fourier/ Laplace transform $t \rightarrow \omega$ etc, seems to lead (at least within certain ‘frequency ranges’) to a localization-type phenomenon for $u^0(x, t)$, somewhat resembling Anderson localization:

$$\left(|Q_1|\omega^2 + iK_1\omega + K_2 + \hat{\mathcal{K}}(\omega) \right) \hat{u}^0 + A^h \Delta \hat{u}^0 = -\hat{f}(x, \omega).$$

Possible interpretation:

The microresonances tend to ‘capture’ the energy at frequencies close to their eigenfrequencies; due to their randomness, a wide range of such eigen-frequencies is represented not allowing the wave to propagate.

Summary:

- A critical high contrast scaling due to “**micro-resonances**” gives rise to numerous “non-classical” effects, described by two-scale limit problems.
- ‘**Partial**’ **degeneracies** often happen in physical problems, and give rise to more of such effects.
- **A general two-scale homogenization theory** can be constructed for such **partial degeneracies**, under a generically held decomposition condition. Resulting limit (homogenized) operator is generically two-scale (and macroscopically ‘non-local’). Strong two-scale resolvent convergence generically holds, implying convergence of semigroups, evolution problems, etc.
- Examples when the key assumption fails, however the conclusions are still held via a two-scale convergence with respect to **measures**, Zhikov 2000.
- From convergence to (high-contrast) error bounds, etc..